Quiz Section 3: Predicate Logic and Inference

Review

Inference Rules

Excluded Middle $\dfrac{}{\therefore\ A \lor \neg A}$

Intro $\land$ $\dfrac{A;\ B}{\therefore\ A \land B}$

Elim $\land$ $\dfrac{A \land B}{\therefore\ A,\ B}$

Intro $\lor$ $\dfrac{A,\ B}{\therefore\ A \lor B}$

Elim $\lor$ $\dfrac{A \lor B;\ \neg A}{\therefore\ B}$

Direct Proof $\dfrac{A \Rightarrow B}{\therefore\ A \to B}$

Modus Ponens $\dfrac{A;\ A \to B}{\therefore\ B}$

Intro $\exists$ $\dfrac{P(c) \text{ for some } c}{\therefore\ \exists x\ P(x)}$

Elim $\forall$ $\dfrac{\forall x\ P(x)}{\therefore\ P(a) \text{ for any } a}$

Intro $\forall$ $\dfrac{\text{Let } a \text{ be arbitrary} \ldots P(a)}{\therefore\ \forall x\ P(x) \qquad \text{(If no other name in } P \text{ depends on } a\text{)}}$

Elim $\exists$ $\dfrac{\exists x\ P(x)}{\therefore\ P(c) \text{ for some } \textit{special } c \qquad \text{list dependencies for } c}$

Task 1 – Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other, if they are equivalent, or neither.

a) $\forall x\ \forall y\ P(x, y)$ $\qquad$ $\forall y\ \forall x\ P(x, y)$

b) $\exists x\ \exists y\ P(x, y)$ $\qquad$ $\exists y\ \exists x\ P(x, y)$

c) $\forall x\ \exists y\ P(x, y)$ $\qquad$ $\forall y\ \exists x\ P(x, y)$

d) $\forall x\ \exists y\ P(x, y)$ $\qquad$ $\exists x\ \forall y\ P(x, y)$

e) $\forall x\ \exists y\ P(x, y)$ $\qquad$ $\exists y\ \forall x\ P(x, y)$

Task 2 – Quantifier Ordering

Let your domain of discourse be a set of `Element` objects given in a list called `Domain`. Imagine you have a predicate `pred`(x, y), which is encoded in the java method `public boolean pred(int x, int y)`. That is you call your predicate `pred` true if and only if the java method returns `true`.

a) Consider the following Java method:

```
public boolean Mystery(Domain D){
    for(Element x : D) {
        for(Element y : D) {
            if(pred(x,y))
                return true;
        }
    }
}
```

`Mystery` corresponds to a quantified formula (for `D` being the domain of discourse), what is that formula?

b) What formula does `mystery2` correspond to

```
public boolean Mystery2(Domain D){
    for(Element x : D) {
        boolean thisXPass = false;
        for(Element y : D) {
            if(pred(x,y))
                thisXPass = true;
        }
        if(!thisXPass)
            return false;
    }
    return true;
}
```

Task 3 – Find the Bug

Each of these inference proofs is incorrect. Identify the line (or lines) which incorrectly apply a law, and explain the error. Then, if the claim is false, give concrete examples of propositions to show it is false. If it is true, write a correct proof.

a) This proof claims to show that given $a \to (b \vee c)$, we can conclude $a \to c$.

1. $a \to (b \vee c)$ [Given]

2.1.	a	[Assumption]
2.2.	$\neg b$	[Assumption]
2.3.	$b \vee c$	[Modus Ponens, from 1 and 2.1]
2.4.	c	[$\vee$ elimination, from 2.2 and 2.3]

2. $a \to c$ [Direct Proof Rule, from 2.1-2.4]

b) This proof claims to show that given $p \to q$ and r, we can conclude $p \to (q \lor r)$.

1. $p \to q$	[Given]
2. r	[Given]
3. $p \to (q \lor r)$	[Intro $\lor$ (1,2)]

c) This proof claims to show that given $p \to q$ and q that we can conclude p

1. $p \to q$	[Given]
2. q	[Given]
3. $\neg p \lor q$	[Law of Implication (1)]
4. p	[Eliminate $\lor$ (2,3)

Task 4 – Formal Proof (Direct Proof Rule)

Show that $\neg t \to s$ follows from $t \lor q$, $q \to r$ and $r \to s$ with a formal proof. Then, translate your proof to English. You can try this problem on Cozy at `https://bit.ly/cse311-23sp-section03-4`

Task 5 – Formal Proof

Show that $\neg p$ follows from $\neg(\neg r \lor t)$, $\neg q \lor \neg s$ and $(p \to q) \land (r \to s)$ with a formal proof. Then, translate your proof to English. You can try this problem on Cozy at `https://bit.ly/cse311-23sp-section03-5`.

Task 6 – All for 1 and One $\forall$

Let the domain of discourse contain only the two object a and b. *For this problem only*, you are allowed to use the following fake equivalence rules

$\forall x\, P(x) \equiv P(a) \land P(b)$	$\forall \to \land$
$\exists x\, P(x) \equiv P(a) \lor P(b)$	$\exists \to \lor$

In this question, Q will stand for some arbitrary fully quantified predicate logic formula.

a) Use a chain of equivalences to show that $Q \land (\exists x\, P(x)) \equiv \exists x\, (Q \land P(x))$.

b) Likewise show that $Q \lor (\exists x\, P(x)) \equiv \exists x\, (Q \lor P(x))$.

c) Are each of these equivalences also true assuming our fake equivalences? Yes or no.

i $Q \land (\forall x\, P(x)) \equiv \forall x\, (Q \land P(x))$

ii $Q \lor (\forall x\, P(x)) \equiv \forall x\, (Q \lor P(x))$.

d) Do the equivalences proven in (a)-(b) hold in every other domain of discourse? Briefly explain why or why not.

Task 7 – Proof, Goof, or Spoof?

For each of the claims below, (1) translate the English proof into a formal proof and (2) say which of the following categories describes the formal proof:

Proof The proof is correct.

Goof The claim is true but the proof is wrong.

Spoof The claim is false.

Finally, (3) if it is a goof, point out the errors in the proof and explain how to correct them, and if it is a spoof, point out the *first* error in the proof and then show that the claim is false by giving a counterexample. (If it is a correct proof, then skip part (3).)

a) Show that r follows from $\neg p$ and $p \leftrightarrow r$.

Proof, Goof, or Spoof: Since we are given that $p \leftrightarrow r$, we know $p \rightarrow r$. We are also given that $\neg p$ holds, so it must be the case that $\neg p \vee (p \vee r)$ holds. This claim is equivalent to $(p \wedge \neg p) \rightarrow r$. Since this last claim starts by assuming both p and $\neg p$, we can infer that this holds with just $\neg p$, giving us $\neg p \rightarrow r$. Since we were given that $\neg p$ holds, we get that r holds.

b) Show that $\exists z\, \forall x\, P(x, z)$ follows from $\forall x\, \exists y\, P(x, y)$.

Proof, Goof, or Spoof: We are given that, for every x, there is some y such that P(x, y) holds. Thus, there must be some object c such that for every x, P(x, c) holds. This shows that there exists an object z such that, for every x, P(x, z) holds.

c) Show that $\exists z\, (P(z) \wedge Q(z))$ follows from $\forall x\, P(x)$ and $\exists y\, Q(y)$.

Proof, Goof, or Spoof: Let z be arbitrary. Since we were given that for every x, P(x) holds, P(z) must hold. Since we were given that there is a y such that Q(y) holds, Q(z) must also hold. From the previous facts, we know that there is some object z such that P(z) and Q(z) hold.

Task 8 – Predicate Logic Formal Proof

Given $\forall x\, (T(x) \rightarrow M(x))$, we wish to prove $(\exists x\, T(x)) \rightarrow (\exists y\, M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof. Then, translate the proof to English.

1. $\forall x\, (T(x) \rightarrow M(x))$ ____________________

    2.1. $\exists x\, T(x)$ ____________________

    2.2. $T(c)$ ____________________

    2.3. $T(c) \rightarrow M(c)$ ____________________

    2.4. $M(c)$ ____________________

    2.5. $\exists y\, M(y)$ ____________________

2. $(\exists x\, T(x)) \rightarrow (\exists y\, M(y))$ ____________________