Reading 03: U-Shaped Proofs and You

One of the most common errors in writing proofs involving algebra is to write them backwards. Remember that in a proof, our goal is to write down a sequence of facts so that each follows from the previous ones (like we did for inference proofs, just in paragraph form). What does it mean to write a proof backwards? The most common way is to write your target first, and then modify your target equation until you derive something "obviously" true. For example, imagine you wanted to show that for all x, $15x + 5 = 5(-3x - 1)$

Invalid Technique: A backwards proof

spoof.

$$\begin{aligned}
15x + 5 &= 5(-3x - 1) & \\
225x^2 + 150x + 25 &= 5^2(-3x - 1)^2 & \text{squaring both sides} \\
225x^2 + 150x + 25 &= 25[-(3x + 1)]^2 & \text{factoring out a negative sign} \\
225x^2 + 150x + 25 &= 25(3x + 1)^2 & (-1)^2 = 1 \\
225x^2 + 150x + 25 &= 25(9x^2 + 6x + 1) & \text{FOILing} \\
9x^2 + 6x + 1 &= 9x^2 + 6x + 1 & \text{Dividing by 25} \\
0 &= 0 & \text{Subtracting the expression from both sides}
\end{aligned}$$

Which is clearly true! So $15x + 5 = 5(-3x - 1)$. □

This proof is incorrect (only one value of x makes this equation true. You might graph the left and right sides of the equation to get a visual sense of what's happening with this particular equation). The problem comes in assuming the equation that we're trying to show is true. We don't know it's true (that's why we're trying to prove it!).

Remember our goal in writing a proof is to convince our reader that a statement is true – if the techniques we use can prove false things, they aren't going to be very convincing techniques! We need to find a way to write algebraic manipulations that can't produce this kind of false statement. There are a few safe options to prove an equation is true without accidentally assuming what you're trying to show.

Let's imagine we were proving $\sum_{i=0}^{n}(i + 1) = \frac{(n+1)(n+2)}{2}$ by induction.

We'd have to show $\sum_{i=0}^{0} i + 1 = \frac{(0+1)(0+2)}{2}$ in our base case and $\left[\sum_{i=0}^{k} = \frac{(k+1)(k+2)}{2}\right] \to \left[\sum_{i=0}^{k+1} = \frac{(k+2)(k+3)}{2}\right]$ in our inductive step. Let's see what each of these would look like using a few safe strategies

Valid Technique 1: Evaluate Both Sides Separately

One option to ensure we don't assume an equation holds is to not assert the equation you're trying to show until the very end of our proof. This method makes it the easiest to make sure you won't accidentally assert what you're trying to show, but it's often harder to discover this kind of proof (for inductive steps at least). The idea for this technique is to only ever "take a small step" and "simplify" both sides of the equation **separately** until they arrive at the same common formula/number.

The Base Case

Evaluating the left-hand-side: $\sum_{i=0}^{0} i + 1 = 0 + 1 = 1$ Evaluating the right-hand-side: $\frac{(0+1)(0+2)}{2} = \frac{2}{2} = 1$

Since both sides of the equation are equal to 1, we know $\sum_{i=0}^{0} i + 1 = \frac{(0+1)(0+2)}{2}$, which is $P(0)$.

The Inductive Case

Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Evaluating the left-hand-side:

$$\sum_{i=0}^{k+1} i+1 = (k+2) + \sum_{i=0}^{k} i+1 = (k+2) + \frac{(k+1)(k+2)}{2}$$

where the last step is an application of the inductive hypothesis.

Evaluating the right-hand-side:

$$\frac{(k+2)(k+3)}{2} = \frac{k^2+5k+6}{2} = \frac{k^2+3k+2}{2} + \frac{2k+4}{2} = \frac{(k+1)(k+2)}{2} + \frac{2(k+2)}{2} = (k+2) + \frac{(k+1)(k+2)}{2}$$

Since both sides of the equation evaluated to the same expression, we know they are equal, and we can conclude $P(k+1)$.

Valid Technique 2: Start from the left, convert to the right

The second option is to transform the left-hand-side of the equation until it looks like the right-hand-side.

The Base Case

$$\sum_{i=0}^{0} i+1 = 0+1 = 1 = \frac{2}{2} = \frac{(0+1)(0+2)}{2}$$

The Inductive Case

Suppose $P(k)$ holds for an arbitrary $k \geq 0$

We will convert the left-hand-side of the equation in $P(k+1)$ into the right-hand-side of the equation

$$\begin{aligned}
\sum_{i=0}^{k+1} i+1 &= (k+2) + \sum_{i=0}^{k} i+1 \\
&= (k+2) + \frac{(k+1)(k+2)}{2} \qquad \text{by IH} \\
&= \frac{2(k+2)}{2} + \frac{(k+1)(k+2)}{2} \\
&= \frac{2k+4}{2} + \frac{k^2+3k+2}{2} \\
&= \frac{k^2+5k+6}{2} \\
&= \frac{(k+3)(k+2)}{2}
\end{aligned}$$

So we have $\sum_{i=0}^{k+1} i+1 = \frac{(k+3)(k+2)}{2}$, which is $P(k+1)$.

Valid Technique 3: Start from a known true equation

The third option is to start from something you **do** know to be true and then perform an algebra operation that keeps it true (e.g. adding 0 to one side, multiplying both sides by 7, or simplifying one side or the other)

Base Case

$$\begin{aligned} 1 &= 1 \\ 0+1 &= \frac{2}{2} \\ \sum_{i=0}^{0}(i+1) &= \frac{(0+1)(0+2)}{2} \end{aligned}$$

The Inductive Case

Suppose $P(k)$ holds for an arbitrary $k \geq 0$

$$\begin{aligned} \sum_{i=0}^{k}(i+1) &= \frac{(k+1)(k+2)}{2} && \text{by IH} \\ \sum_{i=0}^{k}(i+1) + k + 2 &= k+2+\frac{(k+1)(k+2)}{2} && \text{add } k+2 \text{ to both sides} \\ \sum_{i=0}^{k+1}(i+1) &= k+2+\frac{(k+1)(k+2)}{2} && \text{combine extra term into summation on left} \\ \sum_{i=0}^{k+1}(i+1) &= \frac{2k+4}{2}+\frac{(k+1)(k+2)}{2} && \text{find common denominator on right} \\ \sum_{i=0}^{k+1}(i+1) &= \frac{2k+4}{2}+\frac{k^2+3k+2}{2} && \text{FOIL second expression on right} \\ \sum_{i=0}^{k+1}(i+1) &= \frac{k^2+5k+6}{2} && \text{combine on right} \\ \sum_{i=0}^{k+1}(i+1) &= \frac{(k+3)(k+2)}{2} && \text{factor on right} \end{aligned}$$

The final equation is the desired $P(k+1)$.

English issues

Sometimes you want to let your reader know where you're going – in an essay, you'll start with a topic sentence, it's sometimes a good idea to do the same thing in a proof (especially a long one). You have to be careful though, that you aren't asserting what you want to prove has already been proven (or that your reader might think that you are). If you want to tell your reader where you're going, you need to be really explicit about it. In our proofs, we sometimes write "Goal: ⟨Thing to be shown⟩." Another option is to clearly delineate by using future tense "We will now show $P(k+1)$"

Those sentences are not asserting that you've already shown either of those statements, it is giving your reader a heads-up where you're headed.

Which Strategy should I use?

It's up to you! As long as it's valid, you can use it. I often find techniques 1 and 2 very useful in base cases of inductive proofs, while 2 and 3 more useful in inductive steps of inductive proofs (where 3 usually starts from the inductive hypothesis). You can "mix an match" these techniques in an induction proof; just because your base case uses technique 1 doesn't mean you have to in your inductive step.

How do I know?

To check if your proof is going in the correct order, read through it, asking yourself with each assertion (each new sentence or each new $=, \leq, \geq$, etc.) "do I know this already, and will my reader believe it?" If the assertion isn't immediately followed by an explanation, then you've asserted it too early, and some part of the proof might be backwards. Remember the valid proof skeletons! We never start with what we're trying to show.

When you have an equation as a goal, it's good to ask yourself which of the three strategies you're using. If you can't tell, you might end up with some backward elements.

Test Yourself: Forward or Backward?

Are these proofs forward or backward? If they're backward, describe how to fix it.

Proof 1

To prove the base case of a proof that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

$$\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2 - 1 = 2^{0+1} - 1$$

Proof 2

The prove the inductive step in a proof where we suppose $P(k) : \sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= 2^{k+1} - 1 \\
2^{k+1} + \sum_{i=0}^{k} 2^i &= 2 \cdot 2^{k+1} - 1 && \text{break off the last term} \\
\sum_{i=0}^{k} 2^i &= 2^{k+1} - 1 && \text{subtract } 2^{k+1} \text{ from both sides}
\end{aligned}$$

Observe that our last statement is the inductive hypothesis, so $P(k+1)$ holds.

Proof 3

To prove the base case of a proof that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Setting $n = 0$

$$\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$$
$$2^0 = 2 - 1$$
$$1 = 1$$

Solution:

- Proof 1 is fine! Each step is a simple algebra step from the previous and considered already justified as you take it.
- Proof 2 is backwards. If you reverse the order (start from the IH, derive $P(k+1)$), you have a correct proof.
- Proof 3 is backwards. We sometimes call this a U-shaped proof. If you read down the left-hand-sides, cross to the right-hand-side at the bottom, and read your way back up (in a U-shape) you would have a correct proof by strategy 2. But as formatted, this is a backward proof (starting from our goal).