CSE 311: Foundations of Computing

Lecture 26: Cardinality, Uncomputability

Check Edstem/email for message about exam

Last HW out
Due next Friday
- No Late Day
Solns next Saturday

Final Exam
Monday June 5
2:30 or 4:30

Not here
Johnnie 102

- Must choose time for exam by Saturday June 3
- Practice Final etc. out room
- Review from Sunday June 4
Last time: Showing that $L$ is not regular

1. “Suppose for contradiction that some DFA $M$ recognizes $L$.”

2. Consider an INFINITE set $S$ of prefixes (which we intend to complete later). It is imperative that for every pair of strings in our set there is an “accept” completion that the two strings DO NOT SHARE. (You need to come up with $S$.)

3. “Since $S$ is infinite and $M$ has finitely many states, there must be two strings $s_a$ and $s_b$ in $S$ for $s_a \neq s_b$ that end up at the same state of $M$.”

4. Consider appending the (hard) completion $t$ to each of the two strings. (You need to come up with a hard $t$ for $s_a, s_b$)

5. “Since $s_a$ and $s_b$ both end up at the same state of $M$, and we appended the same string $t$, both $s_a t$ and $s_b t$ end at the same state $q$ of $M$. Since $s_a t \in L$ and $s_b t \notin L$, $M$ does not recognize $L$.”

6. “Thus, no DFA recognizes $L$.”
Important Notes

• It is not necessary for our strings $x \cdot z$ with $x \in L$ to produce every possible string in the language
  – we only need to find a small “core” set of strings that must be distinguished by the machine

• It is not true that, if $L$ is irregular and $L \subseteq U$, then $U$ is irregular!
  – we always have $L \subseteq \Sigma^*$ and $\Sigma^*$ is regular!
  – our argument needs different answers: $x \cdot z \in L \iff y \cdot z \in L$
    for $\Sigma^*$, both strings are always in the language

Do not claim in your proof that, because $L \subseteq U$, $U$ is also irregular
Last time: Languages and Representations

All

Context-Free
- e.g. palindromes, balanced parens, \( \{0^n1^n : n \geq 0\} \)

Regular
- DFA
- NFA
- Regex

Finite
- \( \{001, 10, 12\} \)
General Computation
Computers from Thought

Computers as we know them grew out of a desire to avoid bugs in mathematical reasoning.

Hilbert in a famous speech at the International Congress of Mathematicians in 1900 set out the goal to **mechanize all of mathematics**.

In the 1930s, work of Gödel and Turing showed that Hilbert’s program is **impossible**.

- **Gödel’s Incompleteness Theorem**
- **Undecidability of the Halting Problem**

Both of these employ an idea we will see called **diagonalization**.

The ideas are simple but so revolutionary that their inventor Georg Cantor was initially shunned by the mathematical leaders of the time:

- Poincaré referred to them as a “**grave disease infecting mathematics.**”
- Kronecker fought to keep Cantor’s papers out of his journals.

**Full employment for mathematicians and computer scientists!!**
Cardinality

What does it mean that two sets have the same size?
Cardinality

What does it mean that two sets have the same size?
A function $f : A \rightarrow B$ is one-to-one (1-1) if every output corresponds to at most one input; i.e. $f(x) = f(x') \Rightarrow x = x'$ for all $x, x' \in A$.

A function $f : A \rightarrow B$ is onto if every output gets hit; i.e. for every $y \in B$, there exists $x \in A$ such that $f(x) = y$.

1-1 but not onto
**Cardinality**

**Definition:** Two sets $A$ and $B$ have the same **cardinality** if there is a one-to-one correspondence between the elements of $A$ and those of $B$. More precisely, if there is a 1-1 and onto function $f : A \to B$.

The definition also makes sense for infinite sets!
Cardinality

Do the natural numbers and the even natural numbers have the same cardinality?

Yes!

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \ldots \\
0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & \ldots \\
\end{array}
\]

What’s the map \( f : \mathbb{N} \to 2\mathbb{N} \)? \( f(n) = 2n \)
Countable sets

**Definition:** A set is **countable** iff it has the same cardinality as some subset of \( \mathbb{N} \).

Equivalent: A set \( S \) is countable iff there is an **onto** function \( g : \mathbb{N} \rightarrow S \).

Equivalent: A set \( S \) is countable iff we can order the elements:

\[
S = \{x_1, x_2, x_3, \ldots\}
\]
The set $\mathbb{Z}$ of all integers

$\ldots, -3, -2, -1, 0, 1, 2, \ldots$
The set $\mathbb{Z}$ of all integers

\begin{align*}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \ldots \\
0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 & 5 & -5 & 6 & -6 & 7 & -7 & \ldots
\end{align*}
The set $\mathbb{Q}$ of rational numbers

We can’t do the same thing we did for the integers.

Between any two rational numbers there are an infinite number of others.
The set of positive rational numbers

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<tr>
<th>1/1</th>
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</table>

... ... ... ... ...
The set of positive rational numbers

The set of all positive rational numbers is countable.

\[ \mathbb{Q}^+ = \{1/1, 2/1, 1/2, 3/1, 2/2, 1/3, 4/1, 2/3, 3/2, 1/4, 5/1, 4/2, 3/3, 2/4, 1/5, \ldots \} \]

List elements in order of numerator+denominator, breaking ties according to denominator.

Only \( k \) numbers have total of sum of \( k + 1 \), so every positive rational number comes up some point.

The technique is called “dovetailing.”

More generally:

• Put all elements into finite groups
• Order the groups
• List elements in order by group (arbitrary order within each group)
The set of positive rational numbers

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</tbody>
</table>

... ... ... ... ...
The set \( \mathbb{Q} \) of rational numbers

\[ \mathbb{Q}^* \]

\[ 0, \frac{1}{1}, -\frac{1}{1}, \frac{2}{1}, -\frac{2}{1}, \frac{1}{2}, -\frac{1}{2}, -\]
Claim: $\Sigma^*$ is countable for every finite $\Sigma$

Dictionary/Alphabetical/Lexicographical order is bad

- Never get past the A’s
- A, AA, AAA,AAAA,AAAAA,AAAAAA, ....
Claim: $\Sigma^*$ is countable for every finite $\Sigma$

Dictionary/Alphabetical/Lexicographical order is bad
- Never get past the A’s
- A, AA, AAA, AAAA, AAAAA, AAAAAA, ....

Instead, use same “dovetailing” idea, except that we group based on length: only $|\Sigma|^k$ strings of length $k$.

e.g. $\{0,1\}^*$ is countable:

$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, ... \}$
The set of all Java programs is countable

Java programs are just strings in $\Sigma^*$ where $\Sigma$ is the alphabet of ASCII characters.

Since $\Sigma^*$ is countable, so is the set of all Java programs.

More generally, any subset of a countable set is countable: it has same cardinality as an (even smaller) subset of $\mathbb{N}$.
OK OK... Is Everything Countable ?!!
Are the real numbers countable?

Theorem [Cantor]:
The set of real numbers between 0 and 1 is not countable.

Proof will be by contradiction.
Uses a new method called diagonalization.
Real numbers between 0 and 1: \([0,1)\)

Every number between 0 and 1 has an infinite decimal expansion:

\[
\begin{align*}
1/2 & = 0.50000000000000000000000... \\
1/3 & = 0.33333333333333333333333... \\
1/7 & = 0.14285714285714285714285... \\
\pi - 3 & = 0.14159265358979323846264... \\
1/5 & = 0.19999999999999999999999... = 0.20000000000000000000000...
\end{align*}
\]

Representation is unique except for the cases that the decimal expansion ends in all 0’s or all 9’s. We will never use the all 9’s representation.
Proof that \([0,1)\) is not countable

Suppose, for a contradiction, that there is a list of them:

\[
\begin{array}{ll}
  r_1 & 0.50000000... \\
  r_2 & 0.33333333... \\
  r_3 & 0.14285714... \\
  r_4 & 0.14159265... \\
  r_5 & 0.12122122... \\
  r_6 & 0.25000000... \\
  r_7 & 0.71828182... \\
  r_8 & 0.61803394... \\
  \ldots & \ldots \\
\end{array}
\]
Proof that \([0,1)\) is not countable

Suppose, for a contradiction, that there is a list of them:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
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<th>7</th>
<th>8</th>
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<tbody>
<tr>
<td>(r_1)</td>
<td>0.</td>
<td>5</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<tr>
<td>(r_2)</td>
<td>0.</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
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<td>...</td>
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<tr>
<td>(r_3)</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>5</td>
<td>7</td>
<td>1</td>
<td>4</td>
<td>...</td>
</tr>
<tr>
<td>(r_4)</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>9</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>...</td>
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<tr>
<td>(r_5)</td>
<td>0.</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
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<td>1</td>
<td>2</td>
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<tr>
<td>(r_6)</td>
<td>0.</td>
<td>2</td>
<td>5</td>
<td>0</td>
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<tr>
<td>(r_7)</td>
<td>0.</td>
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<td>(r_8)</td>
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</table>
Proof that \([0,1)\) is not countable

Suppose, for a contradiction, that there is a list of them:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
\hline
r_1 & 0. & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
r_2 & 0. & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \ldots \\
r_3 & 0. & 1 & 4 & 2 & 8 & 5 & 7 & 1 & 4 & \ldots \\
r_4 & 0. & 1 & 4 & 1 & 5 & 9 & 2 & 6 & 5 & \ldots \\
r_5 & 0. & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & \ldots \\
r_6 & 0. & 2 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
r_7 & 0. & 7 & 1 & 8 & 2 & 8 & 1 & 8 & 2 & \ldots \\
r_8 & 0. & 6 & 1 & 8 & 0 & 3 & 3 & 9 & 4 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
\]
Proof that $[0,1)$ is not countable

Suppose, for a contradiction, that there is a list of them:

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<th>9</th>
<th>...</th>
<th>...</th>
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<tbody>
<tr>
<td>$r_1$</td>
<td>0.</td>
<td>5</td>
<td>0</td>
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<td>0</td>
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<tr>
<td>$r_2$</td>
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<td>3</td>
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<tr>
<td>$r_3$</td>
<td>0.</td>
<td>1</td>
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<td>8</td>
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<td>7</td>
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<tr>
<td>$r_4$</td>
<td>0.</td>
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<td>1</td>
<td>5</td>
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<tr>
<td>$r_5$</td>
<td>0.</td>
<td>1</td>
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<td>1</td>
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<td>2</td>
<td>1</td>
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<tr>
<td>$r_6$</td>
<td>0.</td>
<td>2</td>
<td>5</td>
<td>0</td>
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<tr>
<td>$r_7$</td>
<td>0.</td>
<td>7</td>
<td>1</td>
<td>8</td>
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<td>8</td>
<td>1</td>
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<tr>
<td>$r_8$</td>
<td>0.</td>
<td>6</td>
<td>1</td>
<td>8</td>
<td>0</td>
<td>3</td>
<td>3</td>
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**Flipping rule:**

Only if the other driver deserves it.
Proof that \([0,1)\) is not countable

Suppose, for a contradiction, that there is a list of them:

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<td>5</td>
<td>0</td>
<td>0</td>
</tr>
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<td><strong>r_2</strong></td>
<td>0.</td>
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<td>3</td>
<td>5</td>
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**Flipping rule:**
- If digit is 5, make it 1.
- If digit is not 5, make it 5.
Proof that \([0,1)\) is not countable

Suppose, for a contradiction, that there is a list of them:

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<td>0.5</td>
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<td>0</td>
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<tr>
<td>r_2</td>
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<td>3</td>
<td>5</td>
<td>3</td>
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<td>1</td>
<td>2</td>
</tr>
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<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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**Flipping rule:**
- If digit is 5, make it 1.
- If digit is not 5, make it 5.

If diagonal element is \(0.x_{11}x_{22}x_{33}x_{44}x_{55} \ldots\) then let’s call the flipped number \(0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \ldots\) \(d = 0.1551555\ldots\)

It cannot appear anywhere on the list!
Proof that \([0,1)\) is not countable

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<td>(r_1)</td>
<td>0.</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(r_2)</td>
<td>0.</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>(r_3)</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>(r_4)</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Flipping rule:
If digit is 5, make it 1.
If digit is not 5, make it 5.

For every \(n \geq 1\):
\(r_n \neq 0.\widehat{x}_{11}\widehat{x}_{22}\widehat{x}_{33}\widehat{x}_{44}\widehat{x}_{55} \ldots\)
because the numbers differ on the \(n\)-th digit!

If diagonal element is \(0.x_{11}x_{22}x_{33}x_{44}x_{55} \ldots\) then let’s call the flipped number \(0.\widehat{x}_{11}\widehat{x}_{22}\widehat{x}_{33}\widehat{x}_{44}\widehat{x}_{55} \ldots\)

It cannot appear anywhere on the list!
Proof that \([0,1)\) is not countable

Suppose, for a contradiction, that there is a list of them:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r_1)</td>
<td>0.5</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(r_2)</td>
<td>0.3</td>
<td>3</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>(r_3)</td>
<td>0.1</td>
<td>4</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>(r_4)</td>
<td>0.1</td>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

Flipping rule:
- If digit is 5, make it 1.
- If digit is not 5, make it 5.

For every \(n \geq 1\):
- \(r_n \neq 0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \ldots\)
- because the numbers differ on the \(n\)-th digit!

So the list is incomplete, which is a contradiction.

Thus the real numbers between 0 and 1 are **not countable**: “uncountable”
The set of all functions $f : \mathbb{N} \rightarrow \{0, \ldots, 9\}$ is uncountable
The set of all functions \( f : \mathbb{N} \rightarrow \{0, \ldots, 9\} \) is uncountable

Supposed listing of all the functions:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
\hline
f_1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots \\
f_2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \ldots & \ldots \\
f_3 & 1 & 4 & 2 & 8 & 5 & 7 & 1 & 4 & \ldots & \ldots \\
f_4 & 1 & 4 & 1 & 5 & 9 & 2 & 6 & 5 & \ldots & \ldots \\
f_5 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & \ldots & \ldots \\
f_6 & 2 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots \\
f_7 & 7 & 1 & 8 & 2 & 8 & 1 & 8 & 2 & \ldots & \ldots \\
f_8 & 6 & 1 & 8 & 0 & 3 & 3 & 9 & 4 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
\]
The set of all functions $f : \mathbb{N} \rightarrow \{0, \ldots, 9\}$ is uncountable

Supposed listing of all the functions:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_2$</td>
<td>3</td>
<td>3(^5)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$f_3$</td>
<td>1</td>
<td>4(^5)</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>$f_4$</td>
<td>1</td>
<td>4</td>
<td>1(^5)</td>
<td>9</td>
</tr>
<tr>
<td>$f_5$</td>
<td>1</td>
<td>2</td>
<td>1(^5)</td>
<td>2</td>
</tr>
<tr>
<td>$f_6$</td>
<td>2(^5)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_7$</td>
<td>7</td>
<td>1</td>
<td>8(^5)</td>
<td>2</td>
</tr>
<tr>
<td>$f_8$</td>
<td>6(^5)</td>
<td>1</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

**Flipping rule:**

- If $f_n(n) = 5$, set $D(n) = 1$.
- If $f_n(n) \neq 5$, set $D(n) = 5$. 

...
The set of all functions $f : \mathbb{N} \to \{0, \ldots, 9\}$ is uncountable

Supposed listing of all the functions:

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... | ...
|---|---|---|---|---|---|---|---|---|---|-----|-----
| $f_1$ | 5 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |     |     
| $f_2$ | 3 | 3 | 5 | 3 | 3 | 3 | 3 | 3 | 3 |     |     
| $f_3$ | 1 | 4 | 2 | 5 | 8 | 5 | 7 | 1 | 4 |     |     
| $f_4$ | 1 | 4 | 1 | 5 | 9 | 2 | 6 | 5 | 5 |     |     
| $f_5$ | 1 | 2 | 1 | 2 | 2 | 5 | 1 | 2 | 2 |     |     
| $f_6$ | 2 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |     |     
| $f_7$ | 7 | 1 | 8 | 2 | 8 | 1 | 8 | 2 | 2 |     |     

Flipping rule:
- If $f_n(n) = 5$, set $D(n) = 1$
- If $f_n(n) \neq 5$, set $D(n) = 5$

For all $n$, we have $D(n) \neq f_n(n)$. Therefore $D \neq f_n$ for any $n$ and the list is incomplete! \( \Rightarrow \) \{ $f$ | $f : \mathbb{N} \to \{0,1, \ldots, 9\}$ \} is not countable
Uncomputable functions

We have seen that:

– The set of all (Java) programs is countable
– The set of all functions $f : \mathbb{N} \rightarrow \{0, \ldots, 9\}$ is not countable

So: There must be some function $f : \mathbb{N} \rightarrow \{0, \ldots, 9\}$ that is not computable by any program!
Recall our language picture

- All
- Java
- Context-Free
  - Binary Palindromes
- Regular
  - 0*
- Finite
  - \{001, 10, 12\}
- DFA
- NFA
- Regex

Red arrow indicates a hierarchy.
Uncomputable functions

Interesting… maybe.

Can we come up with an explicit function that is uncomputable?