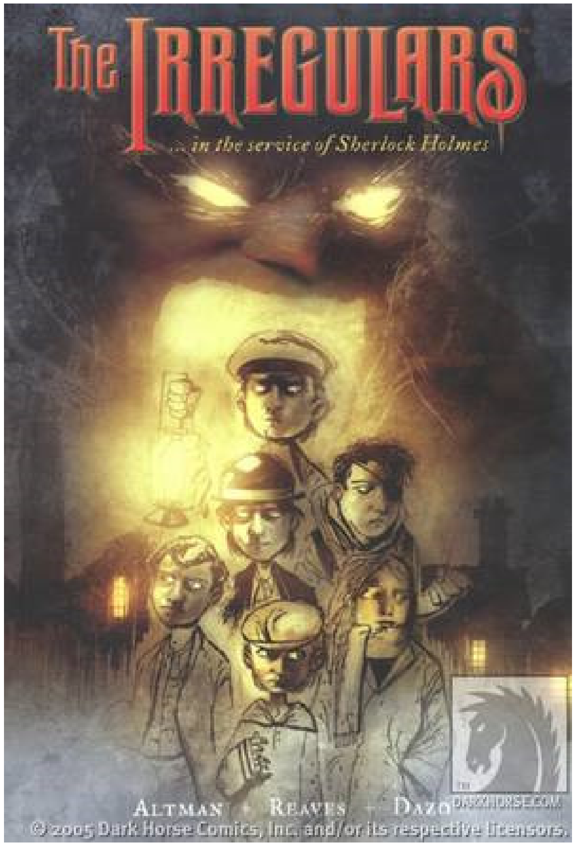
CSE 311: Foundations of Computing

Lecture 25: Languages vs Representations: Limitations of Finite Automata and Regular Expressions



Last time: Algorithms for Regular Languages

We have seen algorithms for

- RE to NFA
- NFA to DFA
- DFA/NFA to RE (not tested)
- DFA minimization

Practice three of these in HW.

(May also be on the final.)

Exponential Blow-up in Simulating Nondeterminism

- In general the DFA might need a state for every subset of states of the NFA
  - Power set of the set of states of the NFA
  - n-state NFA yields DFA with at most 2^n states
  - We saw an example where roughly 2^n is necessary

    "Is the n^{th} char from the end a 1?"

The famous "P=NP?" question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms

Applications of FSMs

- Implementation of regular expression matching in programs like `grep`
- Control structures for sequential logic in digital circuits
- Algorithms for communication and cache-coherence protocols
  – Each agent runs its own FSM
- Design specifications for reactive systems
  – Components are communicating FSMs

Applications of FSMs

- Formal verification of systems
  - Is an unsafe state reachable?
- Computer games
  - FSMs provide worlds to explore
- Minimization algorithms for FSMs can be extended to more general models used in
  - Text prediction
  - Speech recognition

Application of FSMs: Pattern matching

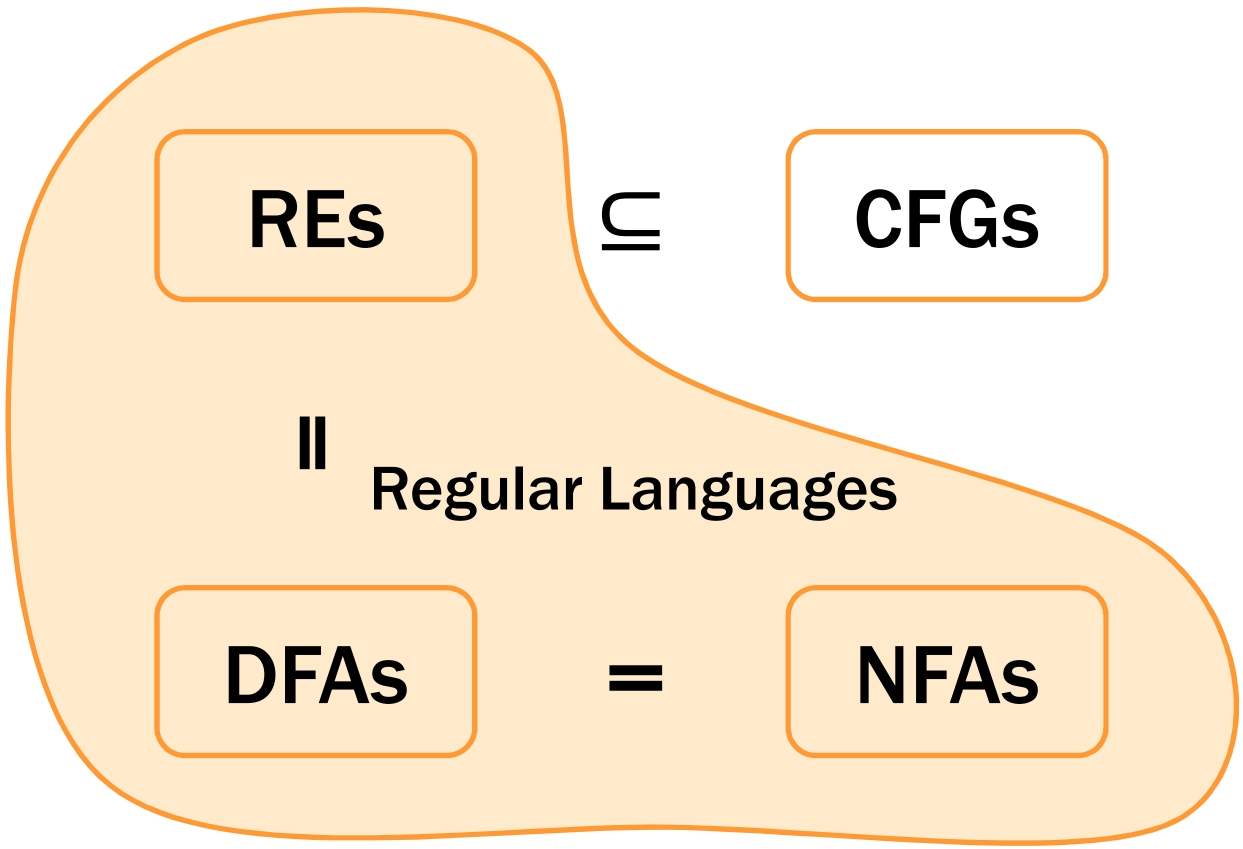
- Given
  - a string s of n characters
  - a pattern p of m characters
  - usually $m \ll n$
- Find
  - all occurrences of the pattern p in the string s

- Obvious algorithm:
  - try to see if p matches at each of the positions in s
    stop at a failed match and try matching at the next
    position: $O(mn)$ running time.

Application of FSMs: Pattern Matching

- With DFAs can do this in $O(m + n)$ time.
- See Extra Credit problem on HW8 for some ideas of how to get to $O(m^2 + n)$ time.

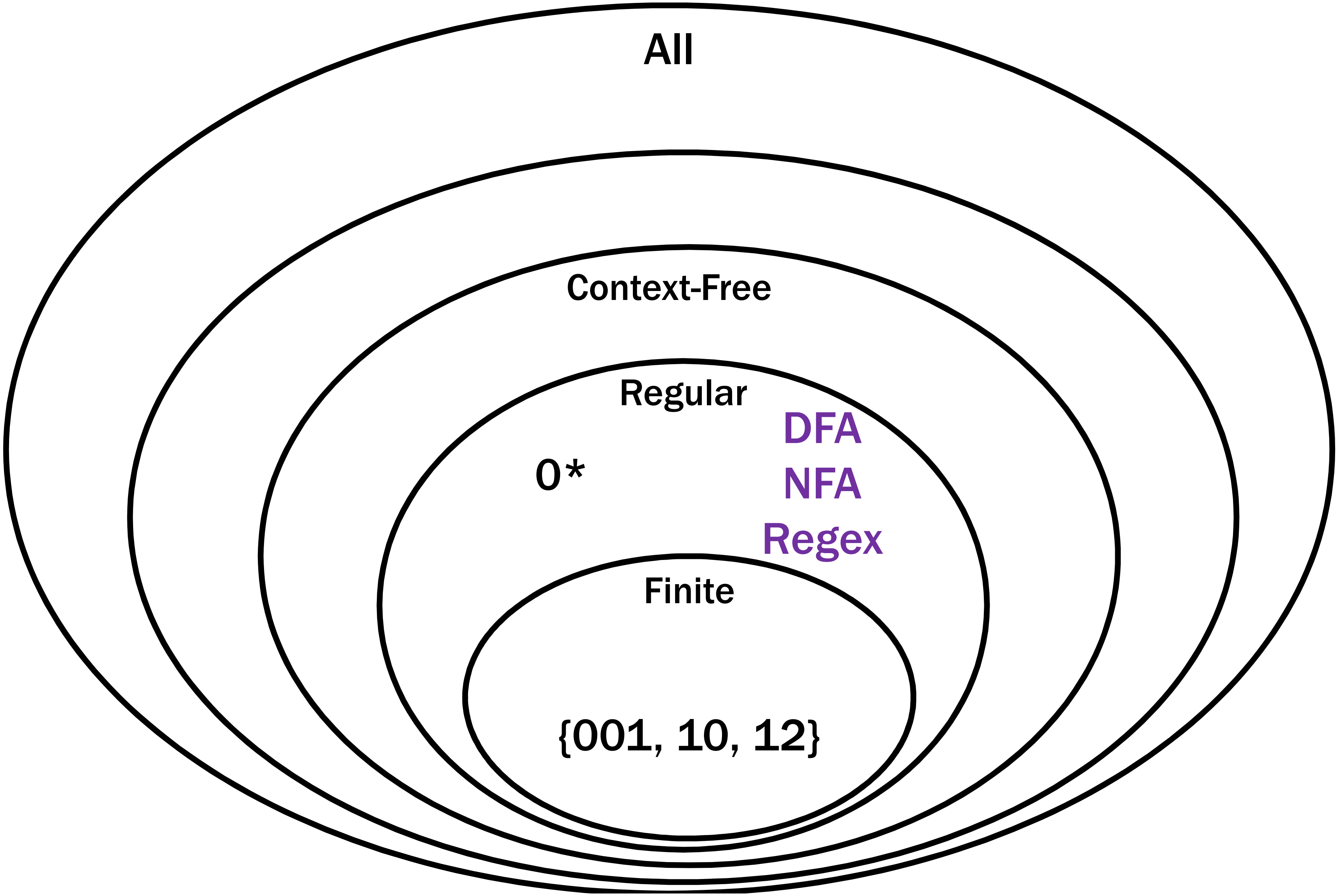
The story so far...



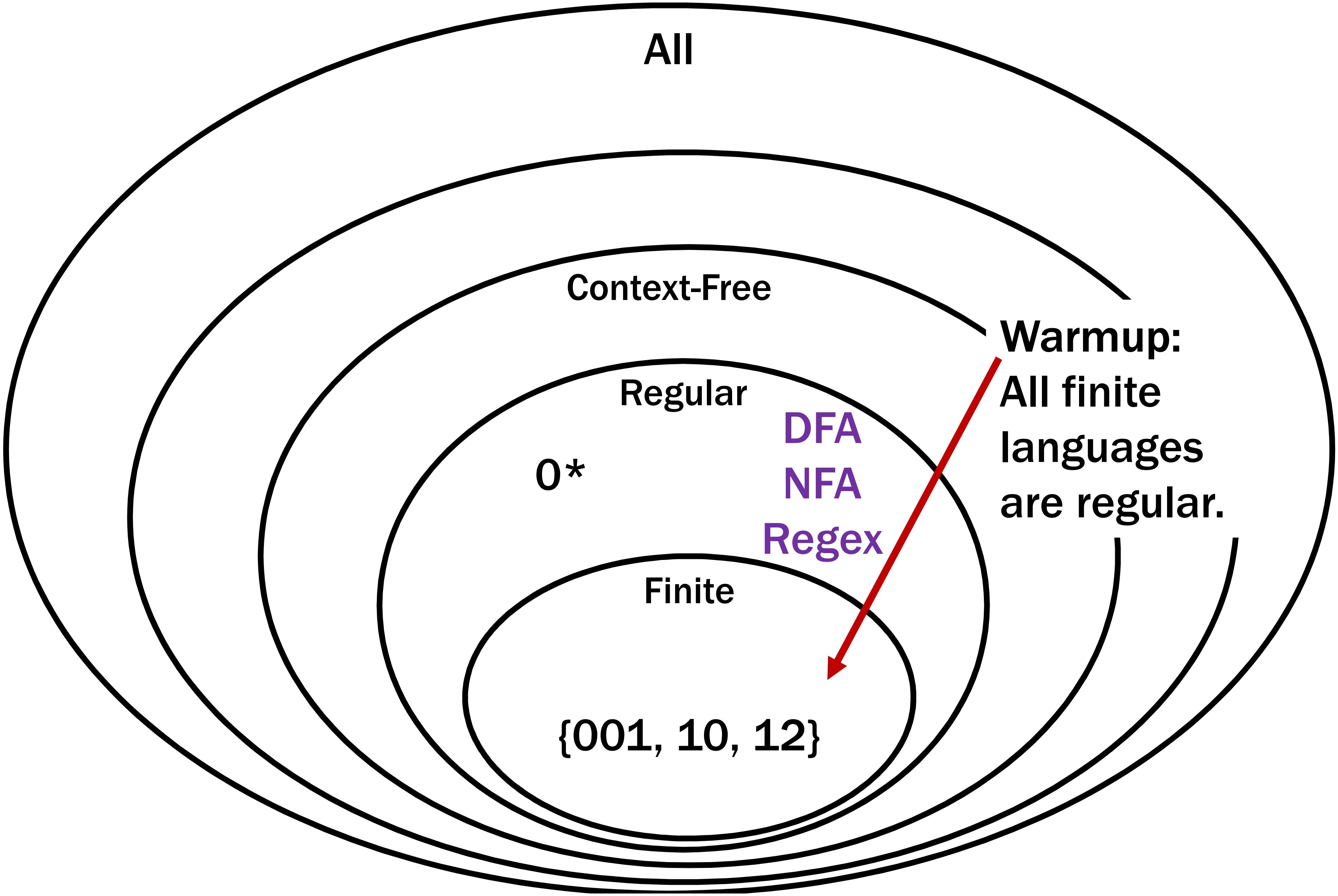
What languages have DFAs? CFGs?

All of them?

Languages and Representations!



Languages and Representations!



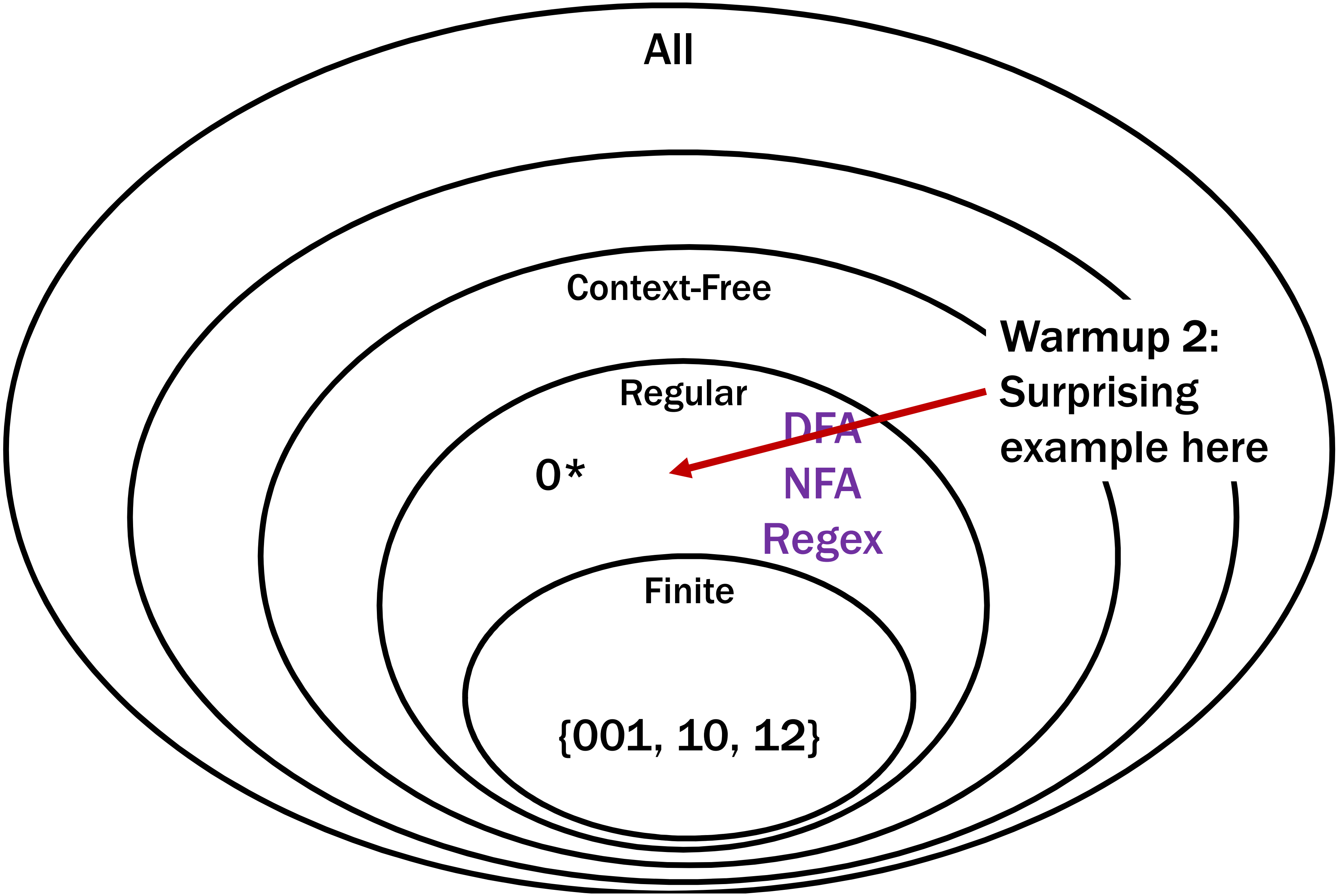
DFAs Recognize Any Finite Language

DFAs Recognize Any Finite Language

Construct a DFA for each string in the language.

Then, put them together using the union construction.

Languages and Machines!



An Interesting Infinite Regular Language

L = {$x \in \{0, 1\}^*$: x has an equal number of substrings 01 and 10}.

L is infinite.

0, 00, 000, ...

L is regular. How could this be?

That seems to require comparing counts...

- easy for a CFG
- but seems hard for DFAs!

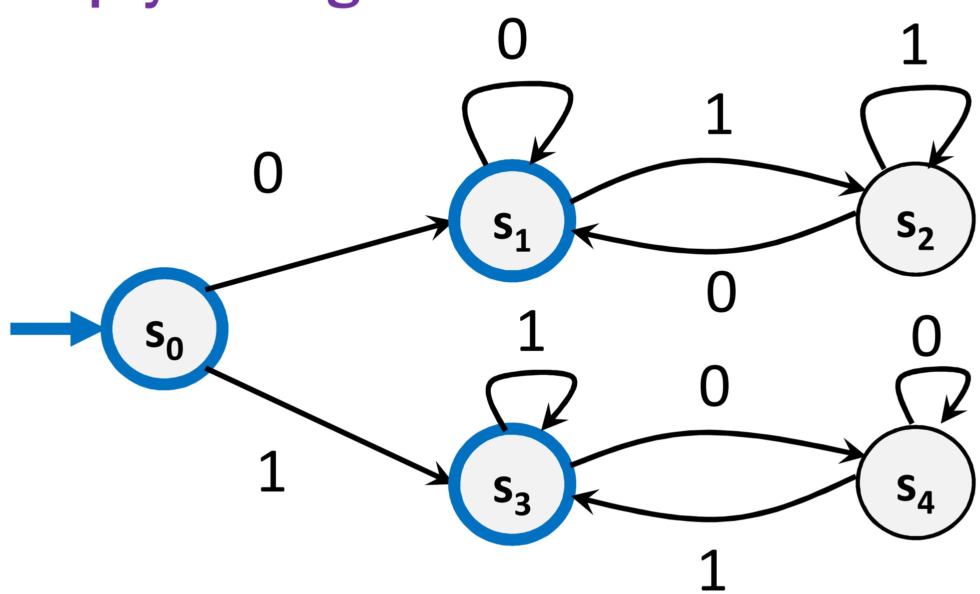
An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

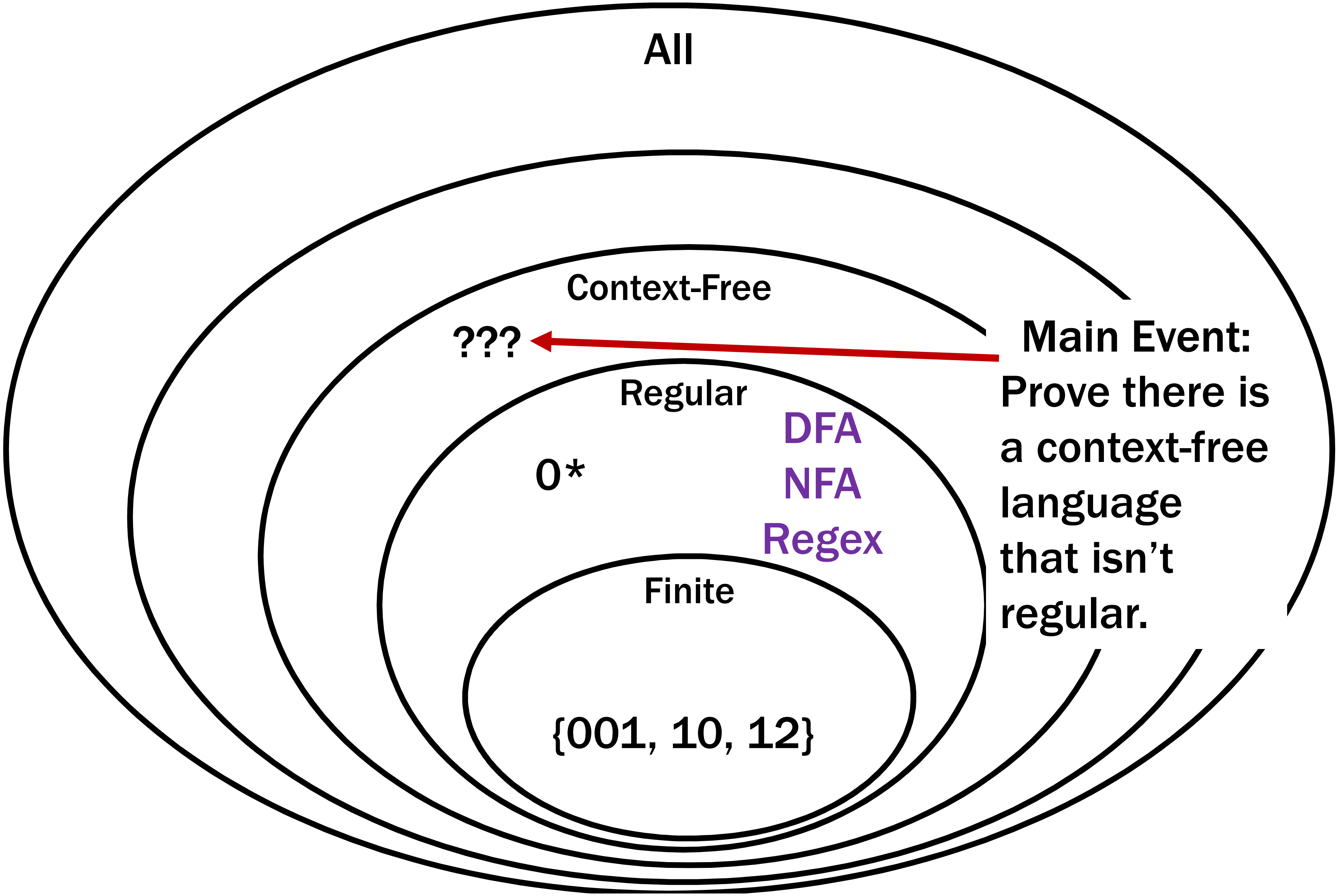
L is infinite.

0, 00, 000, ...

L is regular. How could this be? It is just the set of binary strings that are empty or begin and end with the same character!



Languages and Representations!



Main Event:
Prove there is a context-free language that isn't regular.

The language of “Binary Palindromes” is Context-Free

$$S \to \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$$

Is the language of “Binary Palindromes” Regular ?

Intuition (NOT A PROOF!):

Q: What would a DFA need to keep track of to decide?

A: It would need to keep track of the “first part” of the input in order to check the second part against it

…but there are an infinite # of possible first parts and we only have finitely many states.

Proof idea: any machine that does not remember the entire first half will be wrong for some inputs

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

– Assume (for contradiction) that some DFA (call it M) exists that recognizes B

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How can a DFA fail to recognize B?

- when it accepts or rejects a string it shouldn't.

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that some DFA (call it M) exists that recognizes B
- Our goal is to show that M actually does not recognize B, i.e., it accepts or rejects a string that it shouldn't

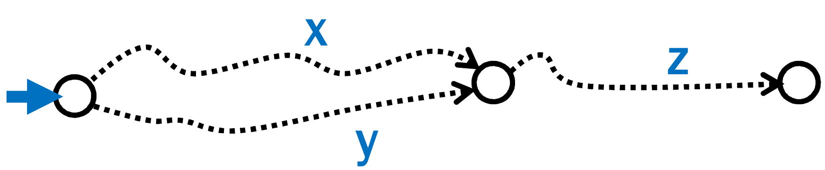
"M recognizes B" AND "M doesn't recognize B", which is a contradiction

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that some DFA (call it M) exists that recognizes B
- We want to show: M accepts or rejects a string it shouldn't.

Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!



$x \bullet z = x_1\, x_2 \ldots x_n\, z_1\, z_2 \ldots z_k$

$y \bullet z = y_1\, y_2 \ldots y_m\, z_1\, z_2 \ldots z_k$

M is correct iff $\forall z \in \Sigma^* \,(x \bullet z \in B \leftrightarrow y \bullet z \in B)$

M is incorrect iff $\exists\, z \in \Sigma^* \,(x \bullet z \in B \not\leftrightarrow y \bullet z \in B)$

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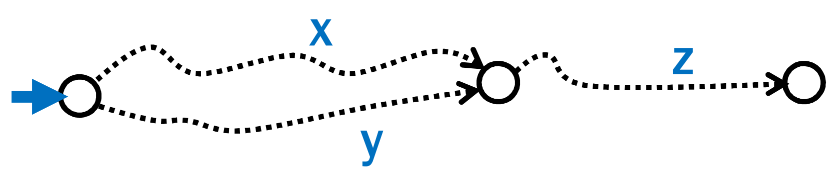

Key Idea 2: Our machine M has a finite number of states which means if we have *infinitely many* strings, two of them must collide!

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- Assume (for contradiction) that some DFA (call it M) exists that recognizes B
- We want to show: M accepts or rejects a string it shouldn't.

We choose an INFINITE set S of prefixes (which we intend to complete later).

x,y z

1______

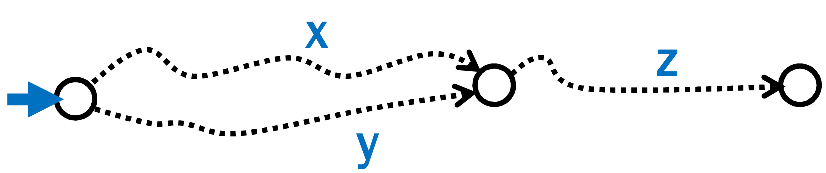
01______

001______

0001______

00001______

....



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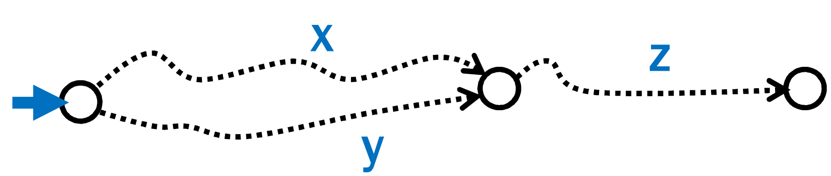
- We want to show: M accepts or rejects a string it shouldn't.

We choose an INFINITE set S of prefixes (which we intend to complete later). It is critical that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.

x,y	z
1	______
01	______
001	______
0001	______
00001	______
....	



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Suppose for contradiction that some DFA, M, recognizes B.

We show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n1 : n \geq 0\}$.

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Since there are finitely many states in M and infinitely many strings in S, there exist strings $0^a1 \in S$ *and* $0^b1 \in S$ *with* $a \neq b$ *that end in the same state of M.*

SUPER IMPORTANT POINT: You do not get to choose what a and b are. Remember, we've just proven they exist…we must take the ones we're given!

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, accepts B.

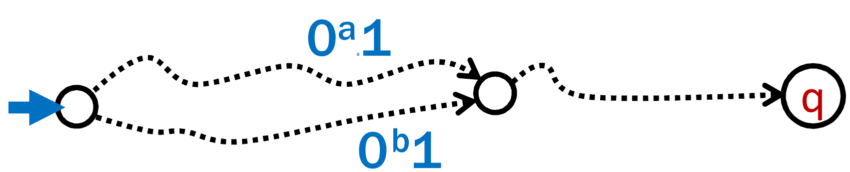
We show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S, there exist strings $0^a1 \in S$ **and** $0^b1 \in S$ **with** $a \neq b$ **that end in the same state of M.**

Now, consider appending 0^a ***to both strings.***

Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!



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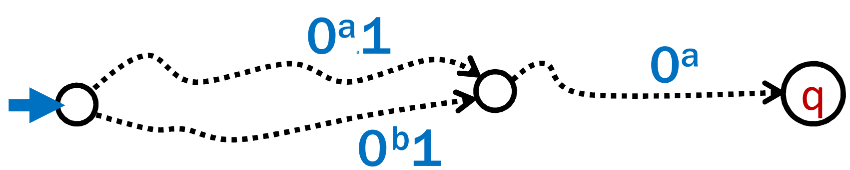

Then, since 0^a1 and 0^b1 end in the same state, 0^a10^a and 0^b10^a also end in the same state, call it q.

But then M makes a mistake: q needs to be an accept state since $0^a10^a \in$ B, but M would accept $0^b10^a \notin$ B which is an error.

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, recognizes B.

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Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n1 : n \geq 0\}$.

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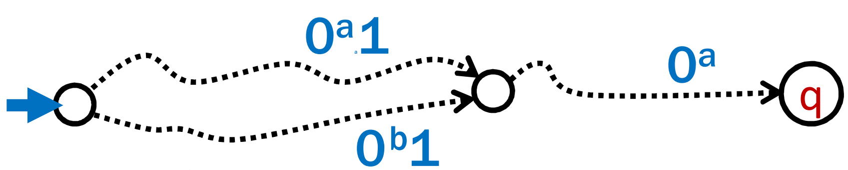

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This is a contradiction since we assumed that M recognizes B. Thus, no DFA recognizes B.

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA M recognizes L."
2. Consider an INFINITE set S of prefxes (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.
3. "Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for $s_a \neq s_b$ that end up at the same state of M."
4. Consider appending the (correct) completion t to each of the two strings.
5. "Since s_a and s_b both end up at the same state of M, and we appended the same string t, both s_at and s_bt end at the same state q of M. Since $s_at \in L$ and $s_bt \notin L$, M does not recognize L."
6. "Thus, no DFA recognizes L."

Prove **A** = $\{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, **M**, recognizes **A**.

Let **S =**

Prove **A** = $\{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, **M**, recognizes **A**.

Let **S** = $\{0^n : n \geq 0\}$. Since **S** is infinite and **M** has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in **M**.

Prove A = $\{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let S = $\{0^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in M.

Consider appending 1^a to both strings.

Prove $A = \{0^n 1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let $S = \{0^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in M.

Consider appending 1^a to both strings.

Note that $0^a 1^a \in A$, but $0^b 1^a \notin A$ since $a \neq b$. But they both end up in the same state of M, call it q. Since $0^a 1^a \in A$, state q must be an accept state but then M would incorrectly accept $0^b 1^a \notin A$ so M does not recognize A.

Thus, no DFA recognizes A.

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, accepts P.

Let S =

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, recognizes P.

Let S = $\{ (^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, $(^a$ and $(^b$ for some $a \neq b$ that end in the same state in M.

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, recognizes P.

Let S = { $(^n$: $n \geq 0$}. Since S is infinite and M has finitely many states, there must be two strings, $(^a$ and $(^b$ for some $a \neq b$ that end in the same state in M.

Consider appending $)^a$ to both strings.

Prove P = {balanced parentheses} is not regular

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Consider appending $)^a$ to both strings.

Note that $(^a)^a \in P$, but $(^b)^a \notin P$ since $a \neq b$. But they both end up in the same state of M, call it q. Since $(^a)^a \in P$, state q must be an accept state but then M would incorrectly accept $(^b)^a \notin P$ so M does not recognize P.

Thus, no DFA recognizes P.

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA M recognizes L."
2. Consider an INFINITE set S of prefixes (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE. (You need to come up with S.)
3. "Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for $s_a \neq s_b$ that end up at the same state of M."
4. Consider appending the (hard) completion t to each of the two strings. (You need to come up with a hard t for s_a, s_b)
5. "Since s_a and s_b both end up at the same state of M, and we appended the same string t, both $s_a t$ and $s_b t$ end at the same state q of M. Since $s_a t \in L$ and $s_b t \notin L$, M does not recognize L."
6. "Thus, no DFA recognizes L."

Fact: This method is optimal

- Suppose that for a language L, the set S is a *largest* set of prefixes with the property that, for every pair $s_a \neq s_b \in S$, there is some string t such that one of $s_a t$, $s_b t$ is in L but the other isn't.
- If S is infinite, then L is not regular
- If S is finite, then the minimal DFA for L has precisely $|S|$ states, one reached by each member of S.

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- If S is infinite, then L is not regular
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Corollary: Our minimization algorithm was correct.

- we separated *exactly* those states for which some t would make one accept and another not accept

Important Notes

- **It is not necessary for our strings xz with $x \in L$ to allow any string in the language**
  - **we only need to find a small "core" set of strings that must be distinguished by the machine**

- **It is not true that, if L is irregular and $L \subseteq U$, then U is irregular!**
  - **we always have $L \subseteq \Sigma^*$ and Σ^* is regular!**
  - **our argument needs different answers:** $xz \in L \not\leftrightarrow yz \in L$

    for Σ^*, both strings are always in the language

Do not claim in your proof that, because $L \subseteq U$, U is also irregular