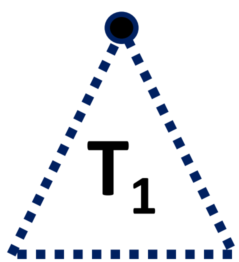
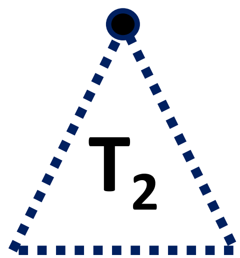
CSE 311: Foundations of Computing

Lecture 18: Strings and Regular Expressions

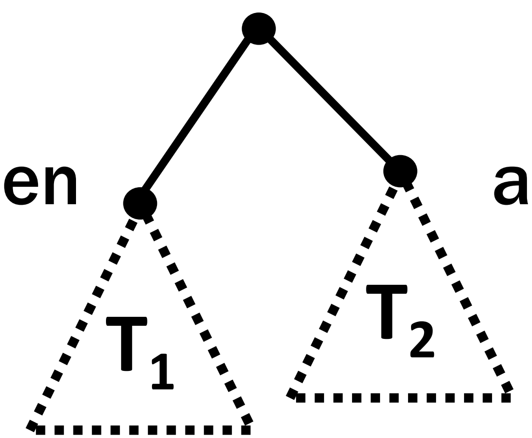
Last time: Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

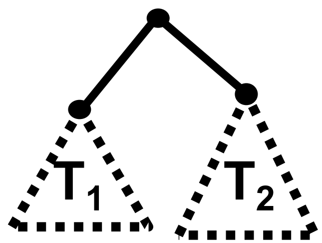
and 

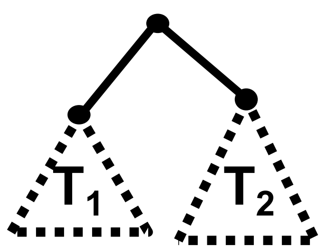
are rooted binary trees,

then 

also is a rooted binary tree.

Defining Functions on Rooted Binary Trees

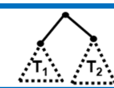
- size(•) := 1
- size (

) := 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)
- height(•) := 0
- height (

) := 1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Claim: **For every rooted binary tree T,** size(**T**) ≤ $2^{\text{height}(\mathbf{T}) + 1} - 1$

1. **Let** P(**T**) **be** "size(T) ≤ $2^{\text{height}(T)+1}-1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
2. **Base Case:** size(•)=1, height(•)=0, **and** $2^{0+1}-1=2^1-1=1$ **so** P(•) **is true.**
3. **Inductive Hypothesis: Suppose that** P($\mathbf{T_1}$) **and** P($\mathbf{T_2}$) **are true for some rooted binary trees $\mathbf{T_1}$ and $\mathbf{T_2}$, i.e.,** size($\mathbf{T}_k$) ≤ $2^{\text{height}(\mathbf{T}_k) + 1} - 1$ **for** k=1,2
4. **Inductive Step:**

Goal: Prove P().

Claim: For every rooted binary tree T, size(T) $\leq 2^{\text{height}(\mathbf{T})+1} - 1$

1. **Let** P(**T**) **be** "size(T) $\leq 2^{\text{height}(T)+1}-1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
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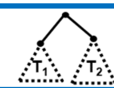
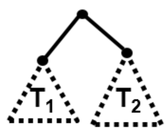
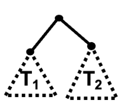
4. **Inductive Step:**

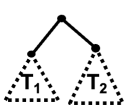
Goal: Prove P().

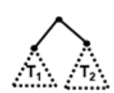
size()

size(•) ::= 1

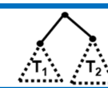
size () ::= 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)

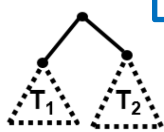
height(•) ::= 0

height () ::= 1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

$\leq 2^{\text{height}(}$  $)^{+1} - 1$

Claim: For every rooted binary tree T, size(T) $\leq 2^{\text{height}(T)+1} - 1$

1. Let P(T) be "size(T) $\leq 2^{\text{height}(T)+1}-1$". We prove P(T) for all rooted binary trees T by structural induction.
2. Base Case: size(•)=1, height(•)=0, and $2^{0+1}-1=2^1-1=1$ so P(•) is true.
3. Inductive Hypothesis: Suppose that $P(T_1)$ and $P(T_2)$ are true for some rooted binary trees T_1 and T_2, i.e., size(T_k) $\leq 2^{\text{height}(T_k)+1}-1$ for k=1,2
4. Inductive Step: **Goal: Prove P().**

By def, size(

) $=1+\text{size}(T_1)+\text{size}(T_2)$

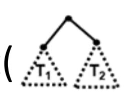
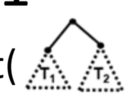
$\leq 1+2^{\text{height}(T_1)+1}-1+2^{\text{height}(T_2)+1}-1$

by IH for T_1 and T_2

$= 2^{\text{height}(T_1)+1}+2^{\text{height}(T_2)+1}-1$

$\leq 2\cdot\max(2^{\text{height}(T_1)+1},2^{\text{height}(T_2)+1})-1$

$= 2(2^{\max(\text{height}(T_1),\text{height}(T_2))+1})-1$

= 2(2^height()) − 1 = 2^height()+1 − 1

which is what we wanted to show.

5. So, the P(T) is true for all rooted binary trees by structural induction.

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ* of *strings* over the alphabet Σ
  - example: {0,1}* is the set of *binary strings*

    0, 1, 00, 01, 10, 11, 000, 001, … and “”

- Σ* is defined recursively by
  - Basis: $\varepsilon \in \Sigma^*$ (ε is the empty string, i.e., “”)
  - Recursive: if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same when read backwards and forwards

Basis:

ε is a palindrome

any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome,

then apa is a palindrome for every $a \in \Sigma$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$len(\varepsilon) := 0$

$len(wa) := len(w) + 1$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon := x$ for $x \in \Sigma^*$

$x \bullet wa := (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R := \varepsilon$

$(wa)^R := a \bullet w^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Number of c's in a string:

$\#_c(\varepsilon) := 0$

$\#_c(wc) := \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) := \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

separate cases for
c vs $a \neq c$

Claim: $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ **for all** $x, y \in \Sigma^*$

Let $P(y)$ **be** "$\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$" .
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case ($y = \varepsilon$): **Let** $x \in \Sigma^*$ **be arbitrary. Then,** $\text{len}(x \bullet \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ **since** $\text{len}(\varepsilon)=0$. **Since** x **was arbitrary,** $P(\varepsilon)$ **holds.**

Inductive Hypothesis: Assume that $P(w)$ **is true for some arbitrary** $w \in \Sigma^*$**, i.e.,** $\text{len}(x \bullet w) = \text{len}(x) + \text{len}(w)$ for all x

Claim: $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ **for all** $x,y \in \Sigma^*$

Does this look familiar?

Let $P(y)$ **be** "$\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x \in$
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural indu**

Base Case ($y = \varepsilon$): **Let** $x \in \Sigma^*$ **be arbitrary. Then,** $\text{len}(x \bullet \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ **since** $\text{len}(\varepsilon)=0$. **Since** x **was arbitrary,** $P(\varepsilon)$ **holds.**

Inductive Hypothesis: Assume that $P(w)$ **is true for some arbitrary** $w \in \Sigma^*$**, i.e.,** $\text{len}(x \bullet w) = \text{len}(x) + \text{len}(w)$ for all x

Inductive Step: **Goal: Show that** $P(wa)$ **is true for every** $a \in \Sigma$

Let $a \in \Sigma$ **and** $x \in \Sigma^*$. **Then** $\text{len}(x \bullet wa) = \text{len}((x \bullet w)a)$ **by def of** $\bullet$

$= \text{len}(x \bullet w)+1$ **by def of len**

$= \text{len}(x)+\text{len}(w)+1$ **by I.H.**

$= \text{len}(x)+\text{len}(wa)$ **by def of len**

Therefore, $\text{len}(x \bullet wa)= \text{len}(x)+\text{len}(wa)$ **for all** $x \in \Sigma^*$**, so** $P(wa)$ **is true.**

So, by induction $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ **for all** $x,y \in \Sigma^*$

Theoretical Computer Science

Languages: Sets of Strings

- Subsets of strings are called *languages*
- Examples:
  - $\Sigma^* =$ All strings over alphabet Σ
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Binary strings with an equal # of 0's and 1's
  - Legal variable names in Java/C/C++
  - Syntactically correct Java/C/C++ programs
  - Valid English sentences

Foreword on Intro to Theory C.S.

- Look at different ways of defining languages
- See which are more expressive than others
  - i.e., which can define more languages

- Later: connect ways of defining languages to different types of (restricted) computers
  - computers capable of recognizing those languages i.e., distinguishing strings in the language from not

- Consequence: computers that recognize more expressive languages are more powerful

Regular Expressions

Regular expressions over Σ

- **Basis:**

  ε is a regular expression (could also include $\varnothing$)

  a is a regular expression for any $a \in \Sigma$

- **Recursive step:**

  If **A** and **B** are regular expressions, then so are:

  A $\cup$ **B**

  AB

  A*

Each Regular Expression is a "pattern"

ε matches only the **empty string**

a matches only the one-character string *a*

A ∪ **B** matches all strings that either **A** matches or **B** matches (or both)

AB matches all strings that have a first part that **A** matches followed by a second part that **B** matches

A* matches all strings that have any number of strings (even 0) that **A** matches, one after another (**ε** ∪ **A** ∪ **AA** ∪ **AAA** ∪ **…**)

Definition of the *language* matched by a regular expression

Examples

001*

0*1*

Examples

001*

{00, 001, 0011, 00111, ...}

0*1*

Any number of 0's followed by any number of 1's

Examples

$(0 \cup 1)\ 0\ (0 \cup 1)\ 0$

$(0^*1^*)^*$

Examples

$(\mathbf{0} \cup \mathbf{1})\,\mathbf{0}\,(\mathbf{0} \cup \mathbf{1})\,\mathbf{0}$

{0000, 0010, 1000, 1010}

$(\mathbf{0}^*\mathbf{1}^*)^*$

All binary strings

Examples

- All binary strings that contain 0110

Examples

- All binary strings that contain 0110

$$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$$

Examples

- All binary strings that contain 0110

$$(0 \cup 1)^*\, 0110\, (0 \cup 1)^*$$

- All binary strings that begin with a string of doubled characters (00 or 11) followed by 01010 or 10001

Examples

- All binary strings that contain 0110

$$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$$

- All binary strings that begin with a string of doubled characters (00 or 11) followed by 01010 or 10001

$$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$$

Regular Expressions in Practice

- Used to define the *tokens* of a programming language
  - legal variable names, keywords, etc.
- Used in `grep`, a program that does pattern matching searches in UNIX/LINUX
- We can use regular expressions in programs to process strings!

Regular Expressions in Java

```
Pattern p = Pattern.compile("a*b");
Matcher m = p.matcher("aaaaab");
boolean b = m.matches();
```

`[01]`	a 0 or a 1	`^` start of string	`$` end of string		
`[0-9]`	any single digit	`\.` period	`\,` comma	`\-` minus	
`.`	any single character				

ab	a followed by b	(**AB**)
`(a|b)`	a or b	($\mathbf{A} \cup \mathbf{B}$)
a`?`	zero or one of a	($\mathbf{A} \cup \varepsilon$)
a`*`	zero or more of a	$\mathbf{A}^*$
a`+`	one or more of a	$\mathbf{AA}^*$

- e.g. `^[\-+]?[0-9]*(\.|\,)?[0-9]+$`

  General form of decimal number e.g. 9.12 or -9,8 (Europe)

Examples

- All binary strings that have an even # of 1's

Examples

- All binary strings that have an even # of 1's

  e.g., $0^*(10^*10^*)^*$

Examples

- All binary strings that have an even # of 1's

  e.g., $0^* (10^*10^*)^*$

- All binary strings that *don't* contain 101

Examples

- All binary strings that have an even # of 1's

e.g., $0^* (10^*10^*)^*$

- All binary strings that *don't* contain 101

e.g., $0^* (1 \cup 000^*)^* 0^*$

at least two 0s between 1s

Limitations of Regular Expressions

- Not all languages can be specified by regular expressions
- Even some easy things like
  - Palindromes
  - Strings with equal number of 0’s and 1’s
- But also more complicated structures in programming languages
  - Matched parentheses
  - Properly formed arithmetic expressions
  - etc.

Context-Free Grammars

- A Context-Free Grammar (CFG) is given by a finite set of substitution rules involving
  - A finite set **V** of *variables* that can be replaced
  - Alphabet **Σ** of *terminal symbols* that can't be replaced
  - One variable, usually **S**, is called the *start symbol*

- The substitution rules involving a variable **A**, written

$$A \to w_1 \mid w_2 \mid \cdots \mid w_k$$

where each w_i is a string of variables and terminals

– that is, $w_i \in (V \cup \Sigma)^*$

How CFGs generate strings

- Begin with start symbol S

- If there is some variable A in the current string you can replace it by one of the w's in the rules for A
  - $A \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$
  - Write this as $xAy \Rightarrow xwy$
  - Repeat until no variables left

- The set of strings the CFG describes are all strings, containing no variables, that can be *generated* in this manner (after a finite number of steps)

Example Context-Free Grammars

Example: $S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$

Example Context-Free Grammars

Example: $S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$

The set of all binary palindromes

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S}0 \mid 1\mathbf{S}1 \mid 0 \mid 1 \mid \varepsilon$

The set of all binary palindromes

Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow \mathbf{0S0} \mid \mathbf{1S1} \mid \mathbf{0} \mid \mathbf{1} \mid \varepsilon$

The set of all binary palindromes

Example: $\mathbf{S} \rightarrow \mathbf{0S} \mid \mathbf{S1} \mid \varepsilon$

0*1*