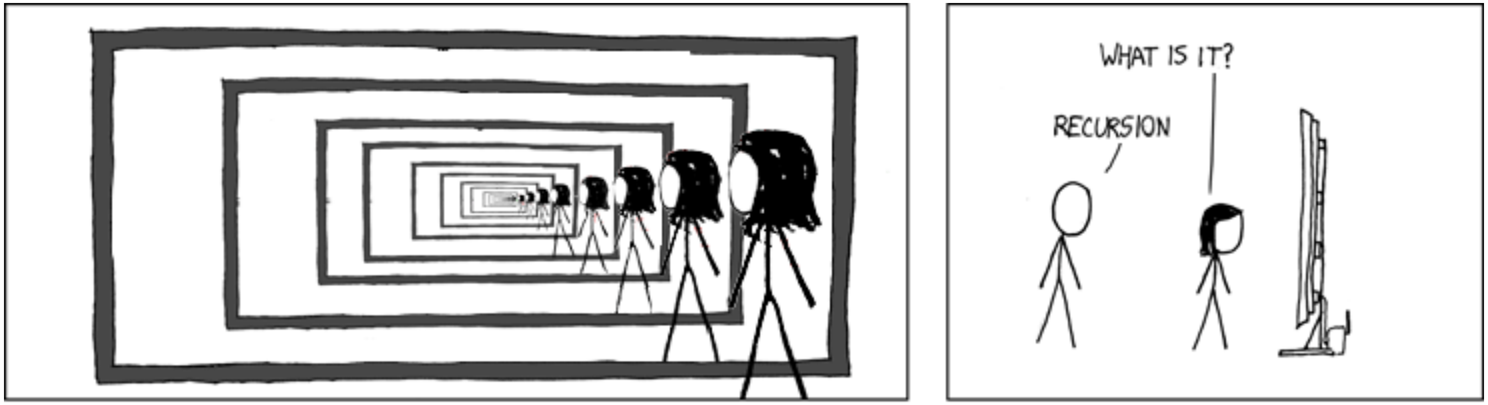
CSE 311: Foundations of Computing

Lecture 16: Recursively Defined Sets & Structural Induction

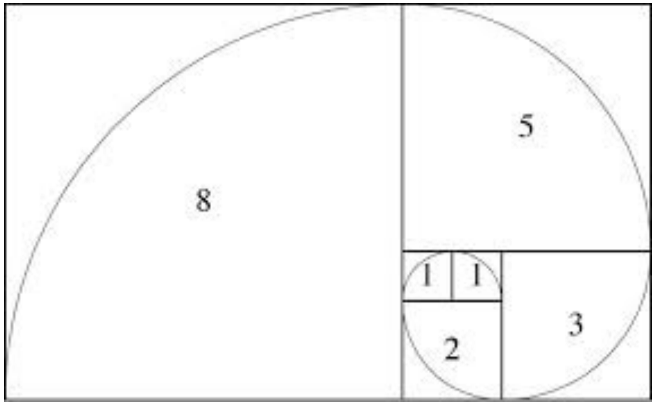


- midterm

Last time: Fibonacci Numbers

$f(0)$

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$



Last Time: Upper Bound $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, we have $f_j < 2^j$ for every integer j from 0 to k.
4. Inductive Step: $\boxed{\text{Goal: Show P(k+1); that is, } f_{k+1} < 2^{k+1}}$

   <u>Case k+1 = 1</u>: Then $f_1 = 1 < 2 = 2^1$ so P(k+1) is true here.

   <u>Case k+1 ≥ 2</u>: Then $f_{k+1} = f_k + f_{k-1}$ by definition

   $< 2^k + 2^{k-1}$ by the IH since $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) is true in this case.

   These are the only cases so P(k+1) follows.
5. Therefore by strong induction,

   $f_n < 2^n$ for all integers $n \geq 0$.

$\boxed{f_0 = 0 \quad f_1 = 1 \\ f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2}$

Inductive Proofs with Multiple Base Cases

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."
2. "Base Cases:" Prove $\boxed{P(b), P(b+1), \ldots, P(c)}$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $\boxed{k \geq c}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

(Strong) Inductive Proofs with Multiple Base Cases

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Cases:" Prove $\boxed{P(b), P(b+1), ..., P(c)}$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $\boxed{k \geq c}$

   $P(j)$ is true for every integer j from b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Original Version

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, we have $f_j < 2^j$ for every integer j from 0 to k.
4. Inductive Step: **Goal: Show P(k+1); that is, $f_{k+1} < 2^{k+1}$**

   Case k+1 = 1: Then $f_1 = 1 < 2 = 2^1$ so P(k+1) is true here.

   Case k+1 ≥ 2: Then $f_{k+1} = f_k + f_{k-1}$ by definition

   $< 2^k + 2^{k-1}$ by the IH since $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) is true in this case.

   These are the only cases so P(k+1) follows.

First case in inductive step didn't need IH

5. Therefore by strong induction,

   $f_n < 2^n$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Multiple Base Case Version

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.

2. Base Cases: $f_0 = 0 < 1 = 2^0$ so P(0) is true.
   $f_1 = 1 < 2 = 2^1$ so P(1) is true.

   Two base cases

   Largest base case

   Smallest base case

3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 1$, we have $f_j < 2^j$ for every integer j from 0 to k.

4. Inductive Step: $\boxed{\text{Goal: Show P(k+1); that is, } f_{k+1} < 2^{k+1}}$

   We have $f_{k+1} = f_k + f_{k-1}$ by definition since $k+1 \geq 2$

   $< 2^k + 2^{k-1}$ by the IH since $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) is true.

5. Therefore, by strong induction, $f_n < 2^n$ for all integers $n \geq 0$.

Two base cases, and two previous values used

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. Let $P(n)$ be "$f_n \geq 2^{n/2 - 1}$". We prove that $P(n)$ is true for all integers $n \geq 2$ by strong induction.

Two base cases, and two previous values used

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Cases:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **holds**
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ **so** P(3) **holds**

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. Let $P(n)$ be "$f_n \geq 2^{n/2 - 1}$". We prove that $P(n)$ is true for all integers $n \geq 2$ by strong induction.
2. Base Cases: $f_2 = f_1 + f_0 = 1$ and $2^{2/2 - 1} = 2^0 = 1$ so $P(2)$ holds
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ so $P(3)$ holds
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 3$, $P(j)$ is true for every integer j from 2 to k.

$$\boxed{P(k+1) \qquad f_{k+1} \geq 2^{(k+1)/2 - 1}}$$

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**".** **We prove that** P(n) **is true for all integers** n ≥ 2 **by strong induction.**
2. **Base Cases:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **holds**
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ **so** P(3) **holds**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** k ≥ 3, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

$$(k+1)/2 - 1 = (k+1)/2 - 2/2 = \frac{(k+1)-2}{2} = \frac{k-1}{2}$$

$$f_{k+1} \geq 2^{(k-1)/2}$$

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Cases:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **holds**
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ **so** P(3) **holds**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 3$, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

   We have $f_{k+1} = f_k + f_{k-1}$ **by definition since** $k+1 \geq 2$

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ **by the IH** 2 **since** $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   so P(k+1) **is true.**
5. **Therefore by strong induction,** $f_n \geq 2^{n/2-1}$ **for all integers** $n \geq 2$.

$$f_0 = 0 \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a,b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

$$r_{n+1} = r_n + r_{n-1}$$

$$r_n = r_{n-1} + r_{n-2}$$

$$\vdots$$

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a,b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid's Algorithm?

We already proved that $f_n \geq 2^{n/2-1}$ so $f_{n+1} \geq 2^{(n-1)/2}$

Therefore: if Euclid's Algorithm takes n steps
for $\gcd(a,b)$ with $a \geq b > 0$
then $a \geq 2^{(n-1)/2}$

so $(n-1)/2 \leq \log_2 a$ or $n \leq 1 + 2\log_2 a$
i.e., # of steps ≤ 1 + twice the # of bits in a.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

An informal way to get the idea: Consider an n **step** gcd **calculation starting with** r_{n+1}=a **and** r_n=b:

$$\begin{aligned}
r_{n+1} &= q_n r_n + r_{n-1} \\
r_n &= q_{n-1} r_{n-1} + r_{n-2} \\
&\ \ \dots \\
r_3 &= q_2 r_2 + r_1 \\
r_2 &= q_1 r_1 + 0
\end{aligned}$$

f_0

For all k ≥ 2, $r_{k-1} = r_{k+1} \bmod r_k$

"Euclid's algorithm is slowest on Fibonacci numbers and it takes only n **steps for** $\gcd(f_{n+1}, f_n)$**"**

Now $r_1 \geq 1$ **and each** q_k **must be** ≥ 1. **If we replace all the** q_K's **by** 1 **and replace** r_1 **by** 1 **, we can only reduce the** r_k's. **After that reduction,** $r_k = f_k$ **for every** k.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n.

Let P(n) **be** "gcd(a,b) **with** $a \geq b > 0$ **takes** n **steps** $\rightarrow a \geq f_{n+1}$" **for all** $n \geq 1$.

Base Case: n=1 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 1 **step.**

By assumption, $a \geq b \geq 1 = f_2$ **so** P(1) **holds.**

n=2 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 2 **steps.**

Then $a = q\,b + r$

$b = q'\,r + 0$ **for** $r \geq 1$.

Since $a \geq b > 0$, **we must have** $q \geq 1$ **and** $b \geq 1$ **so**

$a = qb + r \geq b + r \geq 1+1 = 2 = f_3$ **and** P(2) **holds**

Induction Hypothesis: **Suppose that for some integer** $k \geq 2$, P(j) **is true for all integers** j **s.t.** $1 \leq j \leq k$

Running time of Euclid's algorithm

<u>Induction Hypothesis</u>: Suppose that for some integer $k \geq 2$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: **Goal: if** $\gcd(a,b)$ **with** $a \geq b > 0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2.}$

Since $k \geq 2$, **if** $\gcd(a,b)$ **with** $a \geq b>0$ **takes** $k+1 \geq 3$ **steps, the first** 3 **steps of Euclid's algorithm on** a **and** b **give us**

$$a = q\,b + r$$

k

$$b = q'\,r + r'$$

$$r = q''\,r' + r''$$

k-1

and there are k-2 **more steps after this. Note that this means that the** $\gcd(b, r)$ **takes** k **steps and** $\gcd(r, r')$ **takes** k-1 **steps.**

2

So since $k, k-1 \geq 1$, **by the IH we have** $b \geq f_{k+1}$ **and** $r \geq f_k$.

Running time of Euclid's algorithm

<u>Induction Hypothesis</u>: Suppose that for some integer $k \geq 2$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: $\boxed{\textbf{Goal: if } \gcd(a,b) \textbf{ with } a \geq b>0 \textbf{ takes } k+1 \textbf{ steps, then } a \geq f_{k+2}.}$

Since $k \geq 2$, **if** $\gcd(a,b)$ **with** $a \geq b>0$ **takes** $k+1 \geq 3$ **steps, the first** 3 **steps of Euclid's algorithm on** a **and** b **give us**

$$a = q\,b + r$$
$$b = q'\,r + r'$$
$$r = q''\,r' + r''$$

and there are $k-2$ **more steps after this. Note that this means that the** $\gcd(b, r)$ **takes** k **steps and** $\gcd(r, r')$ **takes** $k-1$ **steps.**

So since $k, k-1 \geq 1$, **by the IH we have** $b \geq f_{k+1}$ **and** $r \geq f_k$.

Also, since $a \geq b$, **we must have** $q \geq 1$.

So $a = q\,b + r \geq b + r \geq f_{k+1} + f_k = f_{k+2}$ **as required.** ■

Last time: Recursive definitions of functions

- $0! = 1$; $(n+1)! = (n+1) \cdot n!$ **for all** $n \geq 0$**.**
- $F(0) = 0$; $F(n+1) = F(n) + 1$ **for all** $n \geq 0$**.**
- $G(0) = 1$; $G(n+1) = 2 \cdot G(n)$ **for all** $n \geq 0$**.**
- $H(0) = 1$; $H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$**.**

Last time: Recursive definitions of functions

- Recursive functions allow general computation
  - saw examples not expressible with simple expressions
- So far, we have considered only simple data
  - inputs and outputs were just integers
- We need general data as well…
  - these will also be described *recursively*
  - will allow us to describe data of real programs

    e.g., strings, lists, trees, expressions, propositions, …
- We'll start simple: sets of numbers

Recursive Definitions of Sets (Data)

$\mathbb{N}$ = integers ≥ 0

Natural numbers

Basis: $0 \in S$

Recursive: **If** $x \in S$, **then** $x+1 \in S$

Even numbers (≥ 0)

Basis: $0 \in S$

Recursive: **If** $x \in S$, **then** $x+2 \in S$

Recursive Definition of Sets

Recursive definition of set S

- **Basis Step:** $0 \in S$
- **Recursive Step: If** $x \in S$**, then** $x + 2 \in S$
- **Exclusion Rule: Every element in** S **follows from the basis step and a finite number of recursive steps.**

We need the exclusion rule because otherwise $S=\mathbb{N}$ **would satisfy the other two parts. However, we won't always write it down on these slides.**

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: **If** $x \in S$, **then** $x+1 \in S$

Even numbers (≥ 0)

Basis: $0 \in S$

Recursive: **If** $x \in S$, **then** $x+2 \in S$

Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, **then** $3x \in S$.

Basis: $(0, 0) \in S$, $(1, 1) \in S$

Recursive: **If** $(n-1, x) \in S$ **and** $(n, y) \in S$,
then $(n+1, x + y) \in S$.

?

(0,0) (1,1)
(2,1) (3,2)
(4,3) (5,5)
(6,8) (7,13)
(8,21) (9,34)

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: **If** $x \in S$, **then** $x+1 \in S$

Even numbers (≥ 0)

Basis: $0 \in S$

Recursive: **If** $x \in S$, **then** $x+2 \in S$

Powers of 3**:**

Basis: $1 \in S$

Recursive: If $x \in S$, **then** $3x \in S$**.**

Basis: $(0, 0) \in S$, $(1, 1) \in S$

Recursive: If $(n-1, x) \in S$ **and** $(n, y) \in S$,

then $(n+1, x + y) \in S$**.**

"Indexed" Fibonacci numbers

$\{(n, f_n) : n \in \mathbb{N}\}$

Last time: Recursive definitions of functions

- Before, we considered only simple data
  - inputs and outputs were just integers

- Proved facts about those functions with **induction**

  $n! \leq n^n$

  $f_n < 2^n$ **and** $f_n \geq 2^{n/2-1}$

- How do we prove facts about functions that work with more complex (recursively defined) data?
  - we need a more sophisticated form of induction

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Structural Induction

Basis: (0, 0) ∈ S, (1, 1) ∈ S
Recursive: If (n-1, x) ∈ S and (n, y) ∈ S,
then (n+1, x + y) ∈ S.

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Structural Induction vs. Ordinary Induction

Structural induction follows from ordinary induction:

Define $Q(n)$ to be "for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true."

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$