CSE 311: Foundations of Computing

Lecture 15: Recursion & Strong Induction
Applications: Fibonacci & Euclid

See Edstem post about 1-1 meetings with TAs not about current HW
Last class: Inductive Proofs In 5 Easy Steps

1. “Let \( P(n) \) be... . We will show that \( P(n) \) is true for all integers \( n \geq b \) by induction.”

2. “Base Case:” Prove \( P(b) \)

3. “Inductive Hypothesis:
   Assume that for some arbitrary integer \( k \geq b \),
   \( P(k) \) is true”

4. “Inductive Step:” Prove that \( P(k + 1) \) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don’t assume \( P(k + 1) \) !!)

5. “Conclusion: \( P(n) \) is true for all integers \( n \geq b \)”
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:
1. Let \( P(n) \) be “Any \( 2^n \times 2^n \) checkerboard with one square removed can be tiled with \( \square \)”. We prove \( P(n) \) for all \( n \geq 1 \) by induction on \( n \).

Base Case \( (n=0) : \ 2^0 \times 2^0 \)
1. Let $P(n)$ be “Any $2^n \times 2^n$ checkerboard with one square removed can be tiled with $\square$ ”. We prove $P(n)$ for all $n \geq 1$ by induction on $n$.

2. Base Case: $n=1$  

3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary $k \geq 1$.  

4. $P(k+1)$:  

\[ 2^{k+1} \times 2^{k+1} \]
1. Let $P(n)$ be “Any $2^n \times 2^n$ checkerboard with one square removed can be tiled with $\square$.”. We prove $P(n)$ for all $n \geq 1$ by induction on $n$.

2. Base Case: $n=1$ 

3. Inductive Hypothesis: Assume $P(k)$ for some arbitrary integer $k \geq 1$
Checkerboard Tiling

1. Let $P(n)$ be “Any $2^n \times 2^n$ checkerboard with one square removed can be tiled with \(\square\)”. We prove $P(n)$ for all $n \geq 1$ by induction on $n$.

2. Base Case: $n=1$

3. Inductive Hypothesis: Assume $P(k)$ for some arbitrary integer $k \geq 1$

4. Inductive Step: Prove $P(k+1)$

Apply IH to each quadrant then fill with extra tile.
Recall: Induction Rule of Inference

Domain: Natural Numbers

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]
\[ \therefore \forall n \ P(n) \]

How do the givens prove \( P(5) \)?

\[ P(0) \rightarrow P(1) \]
\[ P(1) \rightarrow P(2) \]
\[ P(2) \rightarrow P(3) \]
\[ P(3) \rightarrow P(4) \]
\[ P(4) \rightarrow P(5) \]
Recall: Induction Rule of Inference

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]
\[ \therefore \forall n \ P(n) \]

How do the givens prove \( P(5) \)?

We made it harder than we needed to ...

- When we proved \( P(2) \) we knew BOTH \( P(0) \) and \( P(1) \)
- When we proved \( P(3) \) we knew \( P(0) \) and \( P(1) \) and \( P(2) \)
- When we proved \( P(4) \) we knew \( P(0), P(1), P(2), P(3) \)
- etc.

That’s the essence of the idea of Strong Induction.
Strong Induction

\[ P(0) \quad \forall k \left( \forall j \ (0 \leq j \leq k \rightarrow P(j)) \rightarrow P(k + 1) \right) \]

\[ \therefore \ \forall n \ P(n) \]
Strong Induction

\[
P(0) \quad \forall k \left( \forall j \left( 0 \leq j \leq k \rightarrow P(j) \right) \rightarrow P(k + 1) \right) \quad \therefore \forall n P(n)
\]

Strong induction for \( P \) follows from ordinary induction for \( Q \) where

\[
Q(k) := \forall j \left( 0 \leq j \leq k \rightarrow P(j) \right)
\]

Note that \( Q(0) = P(0) \) and \( Q(k + 1) \equiv Q(k) \wedge P(k + 1) \) and \( \forall n Q(n) \equiv \forall n P(n) \)
1. “Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction.”

2. “Base Case:” Prove $P(b)$

3. “Inductive Hypothesis:
   
   Assume that for some arbitrary integer $k \geq b$,
   
   $P(k)$ is true”

4. “Inductive Step:” Prove that $P(k + 1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don’t assume $P(k + 1)$ !!)

5. “Conclusion: $P(n)$ is true for all integers $n \geq b$”
**Strong Inductive Proofs In 5 Easy Steps**

1. “Let \( P(n) \) be... . We will show that \( P(n) \) is true for all integers \( n \geq b \) by **strong** induction.”

2. “Base Case:” Prove \( P(b) \)

3. “Inductive Hypothesis:
   Assume that for some arbitrary integer \( k \geq b \),
   \( P(j) \) is true for every integer \( j \) from \( b \) to \( k \)”

4. “Inductive Step:” Prove that \( P(k + 1) \) is true:
   *Use the goal to figure out what you need.*
   *Make sure you are using I.H. (that \( P(b), \ldots, P(k) \) are true)* and point out where you are using it.
   *(Don’t assume \( P(k + 1) \) !!)*

5. “Conclusion: \( P(n) \) is true for all integers \( n \geq b \)”
Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

\[ 48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \]
\[ 591 = 3 \cdot 197 \]
\[ 45,523 = 45,523 \]
\[ 321,950 = 2 \cdot 5 \cdot 5 \cdot 47 \cdot 137 \]
\[ 1,234,567,890 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 3,607 \cdot 3,803 \]

We use strong induction to prove that a factorization into primes exists, but not that it is unique.
Every integer \( \geq 2 \) is a product of (one or more) primes.
Every integer \( \geq 2 \) is a product of (one or more) primes.

1. Let \( P(n) \) be “n is a product of some list of primes”. We will show that \( P(n) \) is true for all integers \( n \geq 2 \) by strong induction.
Every integer $\geq 2$ is a product of (one or more) primes.

1. Let $P(n)$ be “$n$ is a product of some list of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.

2. Base Case ($n=2$): 2 is prime, so it is a product of (one) prime. Therefore $P(2)$ is true.

3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$ that $P(j)$ is true for all integers $j$ with $2 \leq j \leq k$.

4. Inductive Step: Goal: Show $P(k+1)$; i.e. $k+1$ is a product of primes.
   - Case: $k+1$ is prime: Then by definition $k+1$ is a product of primes.
   - Case: $k+1$ is composite: Then $k+1 = ab$ for some integers $a$ and $b$ where $2 \leq a, b \leq k$.
     By our IH, $P(a)$ and $P(b)$ are true so we have $a = p_1 p_2 \cdots p_m$ and $b = q_1 q_2 \cdots q_n$ for some primes $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n$.
     Thus, $k+1 = ab = p_1 p_2 \cdots p_m q_1 q_2 \cdots q_n$ which is a product of primes.

5. Thus $P(n)$ is true for all integers $n \geq 2$, by induction.
Every integer \( \geq 2 \) is a product of (one or more) primes.

1. Let \( P(n) \) be “\( n \) is a product of some list of primes”. We will show that \( P(n) \) is true for all integers \( n \geq 2 \) by strong induction.

2. Base Case (\( n=2 \)): \( 2 \) is prime, so it is a product of (one) prime. Therefore \( P(2) \) is true.

3. Inductive Hyp: Suppose that for some arbitrary integer \( k \geq 2 \), \( P(j) \) is true for every integer \( j \) between \( 2 \) and \( k \).
Every integer $\geq 2$ is a product of (one or more) primes.

1. Let $P(n)$ be “$n$ is a product of some list of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.

2. Base Case ($n=2$): 2 is prime, so it is a product of (one) prime. Therefore $P(2)$ is true.

3. Inductive Hyp: Suppose that for some arbitrary integer $k \geq 2$, $P(j)$ is true for every integer $j$ between 2 and $k$.

4. Inductive Step:
   
   **Goal:** Show $P(k+1)$; i.e. $k+1$ is a product of primes.
Every integer $\geq 2$ is a product of (one or more) primes.

1. Let $P(n)$ be “$n$ is a product of some list of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.

2. Base Case ($n=2$): $2$ is prime, so it is a product of (one) prime. Therefore $P(2)$ is true.

3. Inductive Hyp: Suppose that for some arbitrary integer $k \geq 2$, $P(j)$ is true for every integer $j$ between 2 and $k$.

4. Inductive Step:
   - **Goal:** Show $P(k+1)$; i.e. $k+1$ is a product of primes.
   - **Case:** $k+1$ is prime: Then by definition $k+1$ is a product of primes.

Since $k \geq 1$, one of these cases must happen and so $P(k+1)$ is true.
Every integer ≥ 2 is a product of (one or more) primes.

1. Let \( P(n) \) be “\( n \) is a product of some list of primes”. We will show that \( P(n) \) is true for all integers \( n \geq 2 \) by strong induction.

2. Base Case (\( n=2 \)): 2 is prime, so it is a product of (one) prime. Therefore \( P(2) \) is true.

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4. Inductive Step:
   
   **Goal:** Show \( P(k+1) \); i.e. \( k+1 \) is a product of primes
   
   **Case:** \( k+1 \) is prime: Then by definition \( k+1 \) is a product of primes
   
   **Case:** \( k+1 \) is composite: Then \( k+1 = ab \) for some integers \( a \) and \( b \) where \( 2 \leq a, b \leq k \).
   
   by IH, \( P(a) \) and \( P(b) \) both true
Every integer $\geq 2$ is a product of (one or more) primes.

1. Let $P(n)$ be “$n$ is a product of some list of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.

2. Base Case ($n=2$): $2$ is prime, so it is a product of (one) prime. Therefore $P(2)$ is true.

3. Inductive Hyp: Suppose that for some arbitrary integer $k \geq 2$, $P(j)$ is true for every integer $j$ between 2 and $k$.

4. Inductive Step:
   
   **Goal:** Show $P(k+1)$; i.e. $k+1$ is a product of primes

   **Case:** $k+1$ is prime: Then by definition $k+1$ is a product of primes

   **Case:** $k+1$ is composite: Then $k+1 = ab$ for some integers $a$ and $b$ where $2 \leq a, b \leq k$. By our IH, $P(a)$ and $P(b)$ are true so we have
   
   $$a = p_1 p_2 \cdots p_r \text{ and } b = q_1 q_2 \cdots q_s$$
   
   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

   Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ which is a product of primes.

   Since $k \geq 2$, one of these cases must happen and so $P(k+1)$ is true.
Every integer $\geq 2$ is a product of (one or more) primes.

1. Let $P(n)$ be “$n$ is a product of some list of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.

2. Base Case ($n=2$): $2$ is prime, so it is a product of (one) prime. Therefore $P(2)$ is true.

3. Inductive Hyp: Suppose that for some arbitrary integer $k \geq 2$, $P(j)$ is true for every integer $j$ between $2$ and $k$.

4. Inductive Step:
   
   **Goal:** Show $P(k+1)$; i.e. $k+1$ is a product of primes

   **Case:** $k+1$ is prime: Then by definition $k+1$ is a product of primes
   
   **Case:** $k+1$ is composite: Then $k+1=ab$ for some integers $a$ and $b$ where $2 \leq a, b \leq k$. By our IH, $P(a)$ and $P(b)$ are true so we have $a = p_1p_2\cdots p_r$ and $b = q_1q_2\cdots q_s$ for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.
   
   Thus, $k+1 = ab = p_1p_2\cdots p_rq_1q_2\cdots q_s$ which is a product of primes. Since $k \geq 2$, one of these cases must happen and so $P(k+1)$ is true.

5. Thus $P(n)$ is true for all integers $n \geq 2$, by strong induction.
Strong Induction is particularly useful when...

...we need to analyze methods that on input $k$ make a recursive call for an input different from $k - 1$.

e.g.: Recursive Modular Exponentiation:
   - For exponent $k > 0$ it made a recursive call with exponent $j = k/2$ when $k$ was even or $j = k - 1$ when $k$ was odd.
public static int FastModExp(int a, int k, int modulus) {

    if (k == 0) {
        return 1;
    } else if ((k % 2) == 0) {
        long temp = FastModExp(a, k/2, modulus);
        return (temp * temp) % modulus;
    } else {
        long temp = FastModExp(a, k-1, modulus);
        return (a * temp) % modulus;
    }

}
Strong Induction is particularly useful when...

...we need to analyze methods that on input $k$ make a recursive call for an input different from $k - 1$.

e.g.: Recursive Modular Exponentiation:
   – For exponent $k > 0$ it made a recursive call with exponent $j = k/2$ when $k$ was even or $j = k - 1$ when $k$ was odd.

We won’t analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.
Recursive definitions of functions

• \(0! = 1;\) \((n + 1)! = (n + 1) \cdot n!\) for all \(n \geq 0\).

• \(F(0) = 0;\) \(F(n + 1) = F(n) + 1\) for all \(n \geq 0\).
  \[\therefore F(n) = n\]

• \(G(0) = 1;\) \(G(n + 1) = 2 \cdot G(n)\) for all \(n \geq 0\).
  \[\therefore G(n) = 2^n\]

• \(H(0) = 1;\) \(H(n + 1) = 2^{H(n)}\) for all \(n \geq 0\).
  \[H(n) = 2^{\text{height}(n)}\]
1. Let $P(n)$ be “$n! \leq n^n$”. We will show that $P(n)$ is true for all integers $n \geq 1$ by induction.

2. Base Case ($n=1$): $1! = 1 \cdot 0! = 1 \cdot 1 = 1 = 1^1$ so $P(1)$ is true.

3. Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 1$. I.e., suppose $k! \leq k^k$. 

Prove $n! \leq n^n$ for all $n \geq 1$
Prove $n! \leq n^n$ for all $n \geq 1$

1. Let $P(n)$ be “$n! \leq n^n$”. We will show that $P(n)$ is true for all integers $n \geq 1$ by induction.

2. Base Case $(n=1)$: $1! = 1 \cdot 0! = 1 \cdot 1 = 1 = 1^1$ so $P(1)$ is true.

3. Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 1$. I.e., suppose $k! \leq k^k$.

4. Inductive Step:

   **Goal:** Show $P(k+1)$, i.e. show $(k+1)! \leq (k+1)^{k+1}$

   \[
   (k+1)! = (k+1)\cdot k!
   \leq (k+1)\cdot k^k
   \leq (k+1)\cdot (k+1)^k
   \leq (k+1)^{k+1}
   \]

   Therefore $P(k+1)$ is true.

5. Thus $P(n)$ is true for all $n \geq 1$, by induction.
More Recursive Definitions

Suppose that $h: \mathbb{N} \to \mathbb{R}$.

Then we have familiar summation notation:

\[
\sum_{i=0}^{0} h(i) = h(0)
\]

\[
\sum_{i=0}^{n+1} h(i) = h(n + 1) + \sum_{i=0}^{n} h(i) \quad \text{for } n \geq 0
\]

There is also product notation:

\[
\prod_{i=0}^{0} h(i) = h(0)
\]

\[
\prod_{i=0}^{n+1} h(i) = h(n + 1) \cdot \prod_{i=0}^{n} h(i) \quad \text{for } n \geq 0
\]

\[
 n! = \prod_{i=1}^{n} i \quad n \geq 1
\]
Fibonacci Numbers

\[ f_0 = 0 \]
\[ f_1 = 1 \]
\[ f_n = f_{n-1} + f_{n-2} \quad \text{for all} \quad n \geq 2 \]
Fibonacci Numbers

\[ f_0 = 0 \]
\[ f_1 = 1 \]
\[ f_n = f_{n-1} + f_{n-2} \] for all \( n \geq 2 \)

A Mathematician's Way* of Converting Miles to Kilometers

\[
\begin{align*}
3 \text{ mi} & \approx 5 \text{ km} \\
5 \text{ mi} & \approx 8 \text{ km} \\
8 \text{ mi} & \approx 13 \text{ km}
\end{align*}
\]

\[ f_n \text{ mi} \approx f_{n+1} \text{ km} \]
Bounding Fibonacci I: \( f_n < 2^n \) for all \( n \geq 0 \)

1. Let \( P(n) \) be “\( f_n < 2^n \)”. We prove that \( P(n) \) is true for all integers \( n \geq 0 \) by strong induction.

\[
\begin{align*}
f_0 &= 0 \\
f_1 &= 1 \\
f_n &= f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2
\end{align*}
\]
Bounding Fibonacci I: \( f_n < 2^n \) for all \( n \geq 0 \)

1. Let \( P(n) \) be “\( f_n < 2^n \)”. We prove that \( P(n) \) is true for all integers \( n \geq 0 \) by strong induction.

2. Base Case: \( f_0=0 < 1= 2^0 \) so \( P(0) \) is true.

3. Inductive Hypothesis: Assume that for some arbitrary integer \( k \geq 0 \), \( P(j) \) is true for every integer \( j \) from 0 to \( k \).

4. Inductive Step: Goal: Show \( P(k+1) \); that is, \( f_{k+1} \leq 2^{k+1} \). By definition, \( f_{k+1} = f_k + f_{k-1} \). By the IH, \( f_{k+1} \leq 2^k + 2^{k-1} \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1} \) so \( P(k+1) \) is true.

5. Therefore by strong induction, \( f_n < 2^n \) for all integers \( n \geq 0 \).
Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let $P(n)$ be “$f_n < 2^n$”. We prove that $P(n)$ is true for all integers $n \geq 0$ by strong induction.

2. Base Case: $f_0 = 0 < 1 = 2^0$ so $P(0)$ is true.

3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, we have $f_j < 2^j$ for every integer $j$ from 0 to $k$. 

4. Inductive Step: Goal: Show $P(k+1)$; that is, $f_{k+1} < 2^{k+1}$.
   - Case $k+1 = 1$: Then $f_1 = 1 \leq 2^1$ so $P(k+1)$ is true here.
   - Case $k+1 \geq 2$: Then $f_{k+1} = f_k + f_{k-1}$ by definition $\leq 2^k + 2^{k-1}$ by the IH $\leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ so $P(k+1)$ is true in this case.

5. Therefore by strong induction, $f_n < 2^n$ for all integers $n \geq 0$. 

$f_0 = 0 \quad f_1 = 1$\
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$
Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let $P(n)$ be “$f_n < 2^n$”. We prove that $P(n)$ is true for all integers $n \geq 0$ by strong induction.

2. Base Case: $f_0 = 0 < 2^0$ so $P(0)$ is true.

3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, we have $f_j < 2^j$ for every integer $j$ from 0 to $k$.

4. Inductive Step: Goal: Show $P(k+1)$; that is, $f_{k+1} < 2^{k+1}$

\[
\begin{align*}
0 \leq f_0 = 0 &< 2^0 = 1 \\
1 \leq f_1 = 1 &< 2^1 = 2 \\
\end{align*}
\]

$$f_n = f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2$$
Bounding Fibonacci I: \( f_n < 2^n \) for all \( n \geq 0 \)

1. Let \( P(n) \) be \( f_n < 2^n \). We prove that \( P(n) \) is true for all integers \( n \geq 0 \) by strong induction.

2. Base Case: \( f_0 = 0 < 1 = 2^0 \) so \( P(0) \) is true.

3. Inductive Hypothesis: Assume that for some arbitrary integer \( k \geq 0 \), we have \( f_j < 2^j \) for every integer \( j \) from 0 to \( k \).

4. Inductive Step: Goal: Show \( P(k+1) \); that is, \( f_{k+1} < 2^{k+1} \)

   Case \( k+1 = 1 \):
   
   \[ f_1 = 1 < 2 = 2^1 \]

   Case \( k+1 \geq 2 \):

\[ f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2 \]
Bounding Fibonacci I: \( f_n < 2^n \) for all \( n \geq 0 \)

1. Let \( P(n) \) be “\( f_n < 2^n \)”. We prove that \( P(n) \) is true for all integers \( n \geq 0 \) by strong induction.

2. Base Case: \( f_0 = 0 < 1 = 2^0 \) so \( P(0) \) is true.

3. Inductive Hypothesis: Assume that for some arbitrary integer \( k \geq 0 \), we have \( f_j < 2^j \) for every integer \( j \) from 0 to \( k \).

4. Inductive Step: Goal: Show \( P(k+1) \); that is, \( f_{k+1} < 2^{k+1} \)

   - **Case** \( k+1 = 1 \): Then \( f_1 = 1 < 2 = 2^1 \) so \( P(k+1) \) is true here.

   - **Case** \( k+1 \geq 2 \):
     \[
     f_{k+1} = f_k + f_{k-1}
     \]
     \[
     f_k < 2^k \quad (\text{IH})
     \]
     \[
     f_{k-1} < 2^{k-1}
     \]
     \[
     f_k + f_{k-1} < 2^k + 2^{k-1} = 2^k + 2^k = 2^{k+1}
     \]

5. Therefore by strong induction, \( f_n < 2^n \) for all integers \( n \geq 0 \).
Bounding Fibonacci I: \( f_n < 2^n \) for all \( n \geq 0 \)

1. Let \( P(n) \) be “\( f_n < 2^n \)”. We prove that \( P(n) \) is true for all integers \( n \geq 0 \) by strong induction.

2. Base Case: \( f_0 = 0 < 1 = 2^0 \) so \( P(0) \) is true.

3. Inductive Hypothesis: Assume that for some arbitrary integer \( k \geq 0 \), we have \( f_j < 2^j \) for every integer \( j \) from 0 to \( k \).

4. Inductive Step: Goal: Show \( P(k+1) \); that is, \( f_{k+1} < 2^{k+1} \)

   - Case \( k+1 = 1 \): Then \( f_1 = 1 < 2 = 2^1 \) so \( P(k+1) \) is true here.
   - Case \( k+1 \geq 2 \): Then \( f_{k+1} = f_k + f_{k-1} \) by definition
     \[ < 2^k + 2^{k-1} \] by the IH since \( k-1 \geq 0 \)
     \[ < 2^k + 2^k = 2 \cdot 2^k \]
     \[ = 2^{k+1} \]

   so \( P(k+1) \) is true in this case.

These are the only cases so \( P(k+1) \) follows.

\[
\begin{align*}
  f_0 &= 0 \\
  f_1 &= 1 \\
  f_n &= f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2
\end{align*}
\]
Bounding Fibonacci I: \( f_n < 2^n \) for all \( n \geq 0 \)

1. Let \( P(n) \) be “\( f_n < 2^n \)”. We prove that \( P(n) \) is true for all integers \( n \geq 0 \) by strong induction.

2. Base Case: \( f_0 = 0 < 1 = 2^0 \) so \( P(0) \) is true.

3. Inductive Hypothesis: Assume that for some arbitrary integer \( k \geq 0 \), we have \( f_j < 2^j \) for every integer \( j \) from \( 0 \) to \( k \).

4. Inductive Step: Goal: Show \( P(k+1) \); that is, \( f_{k+1} < 2^{k+1} \)

   - **Case \( k+1 = 1 \):** Then \( f_1 = 1 < 2 = 2^1 \) so \( P(k+1) \) is true here.

   - **Case \( k+1 \geq 2 \):** Then \( f_{k+1} = f_k + f_{k-1} \) by definition
     \[ < 2^k + 2^{k-1} \text{ by the IH since } k-1 \geq 0 \]
     \[ < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1} \]
     so \( P(k+1) \) is true in this case.

   These are the only cases so \( P(k+1) \) follows.

5. Therefore by strong induction, \( f_n < 2^n \) for all integers \( n \geq 0 \).

\[
\begin{align*}
f_0 &= 0 \\
f_1 &= 1 \\
f_n &= f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2
\end{align*}
\]