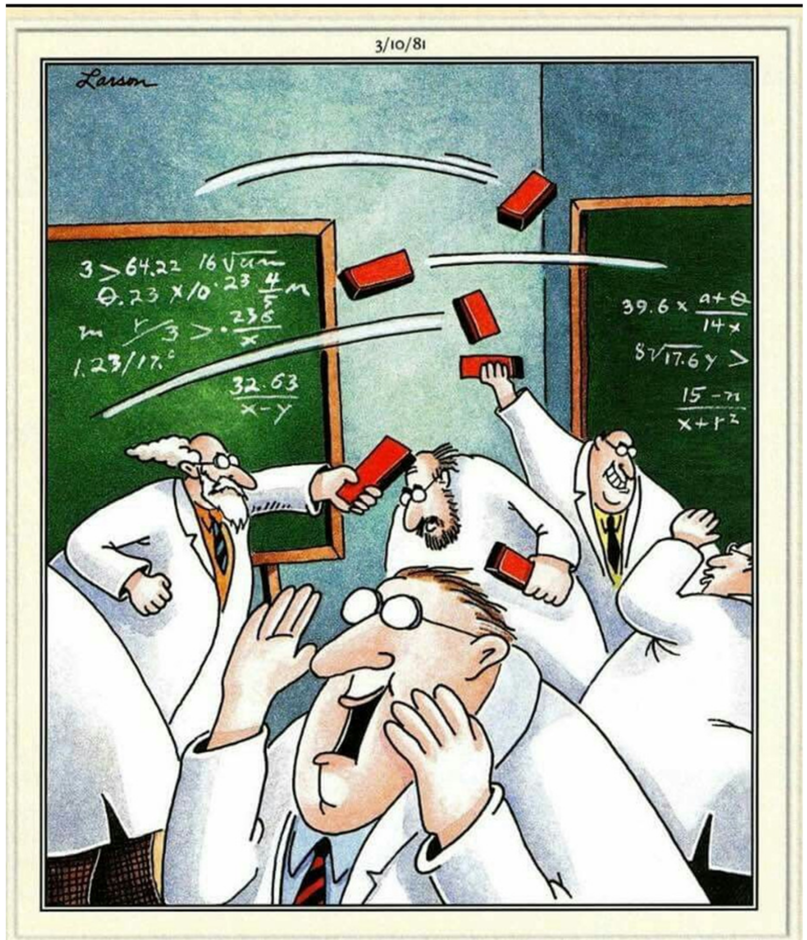
CSE 311: Foundations of Computing

Lecture 13: Set Theory



“Eraser fight!!”

Last class: Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N} = \{0, 1, 2, \ldots\}$
$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48
$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, $\pi, \sqrt{2}$
[n] is the set **{1, 2, …, n}** when **n** is a natural number
$\varnothing$ = **{}** is the **empty set**; the *only* set with no elements

Last class: Definitions

- A and B are *equal* if they have the same elements

$$A = B \; := \; \forall x\, (x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \; := \; \forall x\, (x \in A \rightarrow x \in B)$$

- Notes:

$$(\boldsymbol{A} = \boldsymbol{B}) \equiv (\boldsymbol{A} \subseteq \boldsymbol{B}) \wedge (\boldsymbol{B} \subseteq \boldsymbol{A})$$

$A \supseteq B$ means $B \subseteq A$

$A \subsetneq B$ — $A \subset B$ means $A \subseteq B$ but $A \neq B$

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \ := \ \forall x \, (x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

QUESTIONS

$\varnothing \subseteq A$? ✓
$A \subseteq B$? ✗
$C \subseteq B$? ✓

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \;:=\; \forall x\,(x \in A \rightarrow x \in B)$$

Another way to write domain restriction.

We will use a shorthand for restriction to a set

$$\forall x \in A,\ P(x) \;:=\; \forall x\,(x \in A \rightarrow P(x))$$

Restricting all quantified variables improves *clarity*

Sets & Logic

Building Sets from Predicates

Every set S defines a predicate "$x \in S$".

We can also define a set from a predicate P:

$$S := \{x : P(x)\}$$

S = the set of all x (in some universe U) for which P(x) is true

In other words... $x \in S \leftrightarrow P(x)$

Proofs About Sets

$A := \{x : P(x)\}$

$B := \{x : Q(x)\}$

Suppose we want to prove $A \subseteq B$.

This is a predicate:

$$A \subseteq B := \forall x\, (x \in A \rightarrow x \in B)$$

Typically: use direct proof of the implication

Proofs About Sets

$A \subseteq B := \forall x\,(x \in A \rightarrow x \in B)$

$A := \{x : P(x)\}$

$B := \{x : Q(x)\}$

Let x be an arbitrary elt of A

Prove that $A \subseteq B$ for P(x):= "x>2" **and** Q(x):="x^2>3"

Proof: Let x be an arbitrary object (in the universe).

Suppose that $x \in A$. By definition, this means $P(x)$.

… Therefore $x > 2$ so $x^2 > 4$ which implies $x^2 > 3$.

Thus, we have $Q(x)$. By definition, this means $x \in B$.

Since x was arbitrary, we have shown, by definition, that $A \subseteq B$. ∎

Operations on Sets

Set Operations

$$A \cup B := \{\, x : (x \in A) \lor (x \in B) \}$$ **Union**

$$A \cap B := \{\, x : (x \in A) \land (x \in B) \}$$ **Intersection**

$$A \setminus B := \{\, x : (x \in A) \land (x \notin B) \}$$ **Set Difference**

A = {1, 2, 3}
B = {3, 5, 6}
C = {3, 4}

<u>QUESTIONS</u>

Using A, B, C and set operations, make…

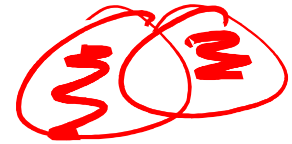
[6] = {1,2,3,4,5,6} $A \cup B \cup C$

{3} = $B \cap C = A \cap B = A \cap C$

{1,2} = $A \setminus B = A \setminus C$

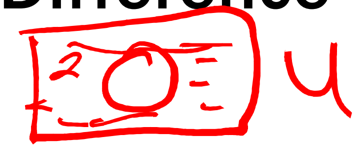
More Set Operations

Δ



$$A \oplus B := \{\, x : (x \in A) \oplus (x \in B) \}$$

Symmetric Difference



$$\bar{A} \;= A^C := \{\, x : x \in U \wedge x \notin A \,\}$$

(with respect to universe U)

A bar or A complement

Complement

A = {1, 2, 3}
B = {1, 2, 4, 6}
Universe:
U = {1, 2, 3, 4, 5, 6}

~~A ⊕ B = {3, 4, 6}~~

Equivalently $x \in \bar{A} \leftrightarrow x \notin A \leftrightarrow \neg(x \in A)$

$A \oplus B$ = {3, 4, 6}
$\bar{A}$ = {4,5,6}

Set Complement

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It's remarkable that as recently as 11 years ago, the sum of all human knowledge could be provided in just two books.

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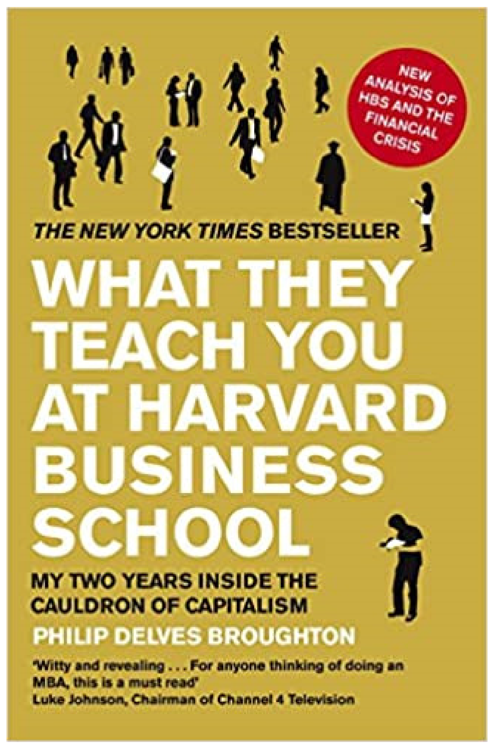
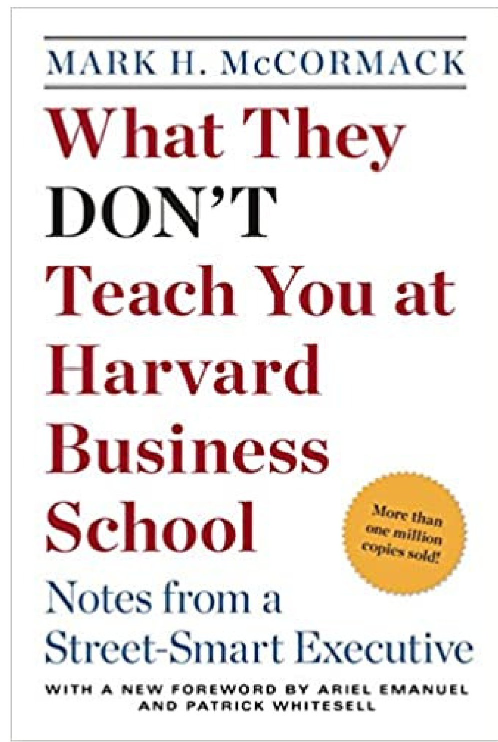




De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

De Morgan's Laws

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Formally, prove $\forall x\, (x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B})$

Proof: Let x be an arbitrary object.

($\Rightarrow$) Suppose that $x \in \overline{A \cup B}$.

…

Thus, we have $x \in \overline{A} \cap \overline{B}$.

Proof technique:
To show C = D show
$x \in$ C $\rightarrow$ $x \in$ D and
$x \in$ D $\rightarrow$ $x \in$ C

De Morgan's Laws

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Formally, prove $\forall x\ (x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B})$

Proof: Let x be an arbitrary object.

($\Rightarrow$) Suppose that $x \in \overline{A \cup B}$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$.

…

Thus, we have $x \in \overline{A} \cap \overline{B}$.

De Morgan's Laws

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Formally, prove $\forall x \ (x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B})$

Proof: Let x be an arbitrary object.

($\Rightarrow$) Suppose that $x \in \overline{A \cup B}$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \lor x \in B)$.

…

Thus, we have $x \in \overline{A} \cap \overline{B}$.

De Morgan's Laws

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Formally, prove $\forall x \ (x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B})$

Proof: Let x be an arbitrary object.

($\Rightarrow$) Suppose that $x \in \overline{A \cup B}$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \lor x \in B)$.

…

Thus, $x \in \overline{A}$ and $x \in \overline{B}$.

Then $x \in \overline{A} \cap \overline{B}$ by the definition of intersection.

De Morgan's Laws

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$
Formally, prove $\forall x \ (x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B})$

Proof: Let x be an arbitrary object.

($\Rightarrow$) Suppose that $x \in \overline{A \cup B}$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(\underbrace{x \in A}_{p} \vee \underbrace{x \in B}_{q})$.

… $\neg(x \in A) \wedge \neg(x \in B)$ ✓ De Morgan

Thus, $\neg(\underbrace{x \in A}_{p})$ and $\neg(\underbrace{x \in B}_{q})$, so $x \in \overline{A}$ and $x \in \overline{B}$ by the definition of complement, and then $x \in \overline{A} \cap \overline{B}$ by the definition of intersection.

De Morgan's Laws

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Formally, prove $\forall x\ (x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B})$

Proof: Let x be an arbitrary object.

($\Rightarrow$) Suppose that $x \in \overline{A \cup B}$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$, or equivalently $\neg(x \in A) \wedge \neg(x \in B)$ by De Morgan's law. Thus, we have $x \in \overline{A}$ and $x \in \overline{B}$ by the definition of complement, and then $x \in \overline{A} \cap \overline{B}$ by the definition of intersection.

To show C = D show
$x \in$ C $\rightarrow x \in$ D and
$x \in$ D $\rightarrow x \in$ C

De Morgan's Laws

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Formally, prove $\forall x \ (x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B})$

Proof: Let x be an arbitrary object.

($\Rightarrow$) Suppose that $x \in \overline{A \cup B}$.... Then, $x \in \overline{A} \cap \overline{B}$.

($\Leftarrow$) Suppose that $x \in \overline{A} \cap \overline{B}$. Then, by the definition of intersection, we have $x \in \overline{A}$ and $x \in \overline{B}$. That is, we have $\neg(x \in A) \wedge \neg(x \in B)$, which is equivalent to $\neg(x \in A \vee x \in B)$ by De Morgan's law. The last is equivalent to $\neg(x \in A \cup B)$, by the definition of union, so we have shown $x \in \overline{A \cup B}$, by the definition of complement. $\blacksquare$

Proofs About Set Equality

A lot of *repetitive* work to show $\rightarrow$ and $\leftarrow$.

Do we have a way to prove $\leftrightarrow$ directly?

Recall that $P \underset{T}{\equiv} Q$ and $(P \leftrightarrow Q) \equiv T$ are the same

We can use an equivalence chain to prove that a biconditional holds.

De Morgan's Laws

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Formally, prove $\forall x \ (x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B})$

Proof: Let x be an arbitrary object.

The stated biconditional holds since:

$$
\begin{aligned}
x \in \overline{A \cup B} &\equiv \neg(x \in A \cup B) && \text{Def of Comp} \\
&\equiv \neg(x \in A \vee x \in B) && \text{Def of Union} \\
&\equiv \neg(x \in A) \wedge \neg(x \in B) && \text{De Morgan} \\
&\equiv x \in \overline{A} \wedge x \in \overline{B} && \text{Def of Comp} \\
&\equiv x \in \overline{A} \cap \overline{B} && \text{Def of } \cancel{\text{Union}} \text{ Intersection}
\end{aligned}
$$

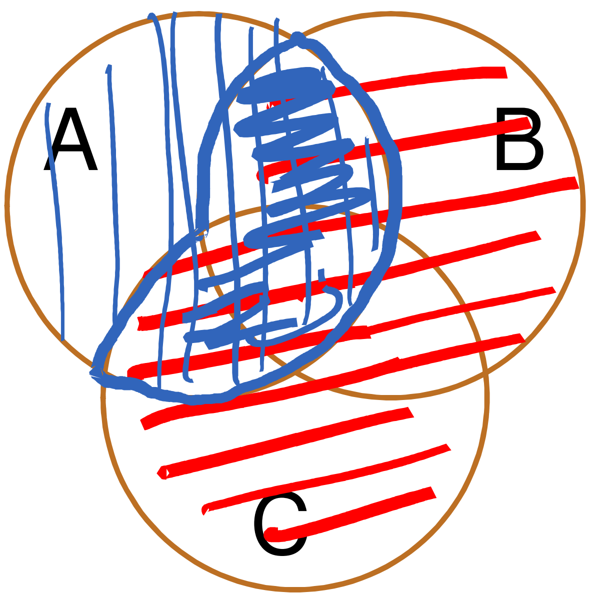
$\blacksquare$

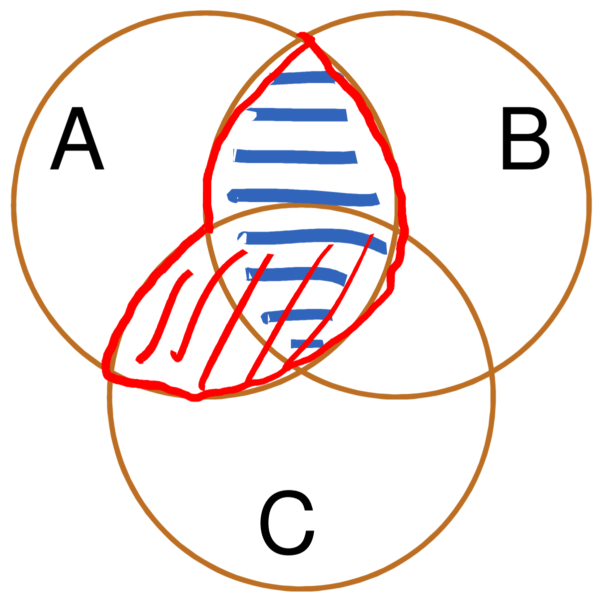
Chains of equivalences are often easier to read like this rather than as English text

Since x was arbitrary, we have shown the sets are equal.

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





It's Propositional Logic Again!

Meta-Theorem: Translate any Propositional Logic equivalence into "=" relationship between sets by replacing $\cup$ with $\vee$, $\cap$ with $\wedge$, and complement with $\neg$.

"Proof": Let x be an arbitrary object.

The stated bi-condition holds since:

$x \in$ left side $\equiv$ replace set ops with propositional logic

$\equiv$ apply Propositional Logic equivalence

$\equiv$ replace propositional logic with set ops

$\equiv x \in$ right side

Since x was arbitrary, we have shown the sets are equal. ■

It’s Boolean Algebra Again!

- Usual notation used in circuit design
- Boolean algebra
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { ’ }
  - such that the following axioms hold:

+ is ∪ ∨
• is ∩ ∧
0 is ∅ F
1 is universe T
A' is $\overline{A}$ ¬

For any a, b, c in B:

1. closure:	a + b is in B	a • b is in B
2. commutativity:	a + b = b + a	a • b = b • a
3. associativity:	a + (b + c) = (a + b) + c	a • (b • c) = (a • b) • c
4. distributivity:	a + (b • c) = (a + b) • (a + c)	a • (b + c) = (a • b) + (a • c)
5. identity:	a + 0 = a	a • 1 = a
6. complementarity:	a + a’ = 1	a • a’ = 0
7. null:	a + 1 = 1	a • 0 = 0
8. idempotency:	a + a = a	a • a = a
9. involution:	(a’)’ = a	

Note on Proofs of Set Equality

Even though it was overly tedious in the De Morgan case…

… the best strategy for proving other cases of set equality $\boldsymbol{A = B}$ is often:

Let x be an arbitrary object.

Show $A \subseteq B$: Assume that $x \in A$ and show that $x \in B$

Show $B \subseteq A$: Assume that $x \in B$ and show that $x \in A$

Power Set

Note $\forall B\,(B \in \mathcal{P}(A) \leftrightarrow B \subseteq A)$

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) := \{B : B \subseteq A\}$$

3

- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$\mathcal{P}$(Days)=? $\{\emptyset, \{M\}, \{W\}, \{F\}, \{M,W\}, \{M,F\}, \{W,F\}, \{M,W,F\}\}$ 8

$\mathcal{P}(\emptyset)$=? $\{\emptyset\}$ $\neq \emptyset$

# of elements $|A| = k$ # of elements $|\mathcal{P}(A)| = 2^k$

Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) := \{B : B \subseteq A\}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$\mathcal{P}(\text{Days})=\{\{M,W,F\},\{M,W\},\{M,F\},\{W,F\},\{M\},\{W\},\{F\},\varnothing\}$

$\mathcal{P}(\varnothing)=?$

Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) := \{B : B \subseteq A\}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$$\mathcal{P}(\text{Days})=\left\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\right\}$$

$$\mathcal{P}(\varnothing)=\{\varnothing\} \neq \varnothing$$

Cartesian Product

$$A \times B := \{x : \exists a \in A, \exists b \in B\ (x = (a, b))\ \}$$

$\{(a,b) : a \in A \text{ and } b \in B\}$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}.

Cartesian Product

always false for $B = \emptyset$

$$A \times B := \{x : \exists a \in A, \exists b \in B\ (x = (a, b))\ \}$$

$\emptyset$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}.

What is $A \times \emptyset$? $= \emptyset$ no pairs

Cartesian Product

$$A \times B := \{x : \exists a \in A, \exists b \in B\ (x = (a, b))\ \}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}.

$$\boldsymbol{A} \times \emptyset = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A}\ \wedge\ \boldsymbol{b} \in \emptyset\} = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A}\ \wedge\ \mathbf{F}\} = \emptyset$$

Russell's Paradox

$$S := \{x : x \notin x\}$$

Suppose that $S \in S$**...**

Russell's Paradox

$\forall x\,(x \in S \leftrightarrow x \notin x)$

$$S := \{x : x \notin x\}$$

Suppose that $S \in S$. Then, by the definition of S, $S \notin S$, but that's a contradiction.

Suppose that $S \notin S$. Then, by the definition of S, $S \in S$, but that's a contradiction too.

This is reminiscent of the truth value of the statement "This statement is false."

need to pick a universe first

Representing Sets Using Bits

- Suppose that universe U is $\{1,2,\ldots,n\}$
- Can represent set $B \subseteq U$ as a vector of bits:

$$b_1 b_2 \ldots b_n \text{ where } \begin{array}{l} b_i = 1 \text{ when } i \in B \\ b_i = 0 \text{ when } i \notin B \end{array}$$

  – Called the *characteristic vector* of set ~~B~~

- Given characteristic vectors for A and B

  What is characteristic vector for $A \cup B$? $A \cap B$?

  $\overline{A}$?

  flip bits

$A \cap B$ 1 0 0 0 0 1 0 0 bitwise AND

A 1 1 0 1 0 1 0 1

B 1 0 1 0 1 1 1 0 0 bitwise OR

$A \cup B$ 1 1 1 1 1 1 1 0 1

Bitwise Operations

bitwise {1, 2, 3, ..., 8}

1 2 3 4 5 6 7 8

$$\begin{array}{rl} & 01101101 \\ \vee & \underline{00110111} \\ & 01111111 \end{array}$$

Java: `z=x|y`

$$\begin{array}{rl} & 00101010 \\ \wedge & \underline{00001111} \\ & 00001010 \end{array}$$

Java: `z=x&y`

$$\begin{array}{rl} & 01101101 \\ \oplus & \underline{00110111} \\ & 01011010 \end{array}$$

001

Java: `z=x^y`

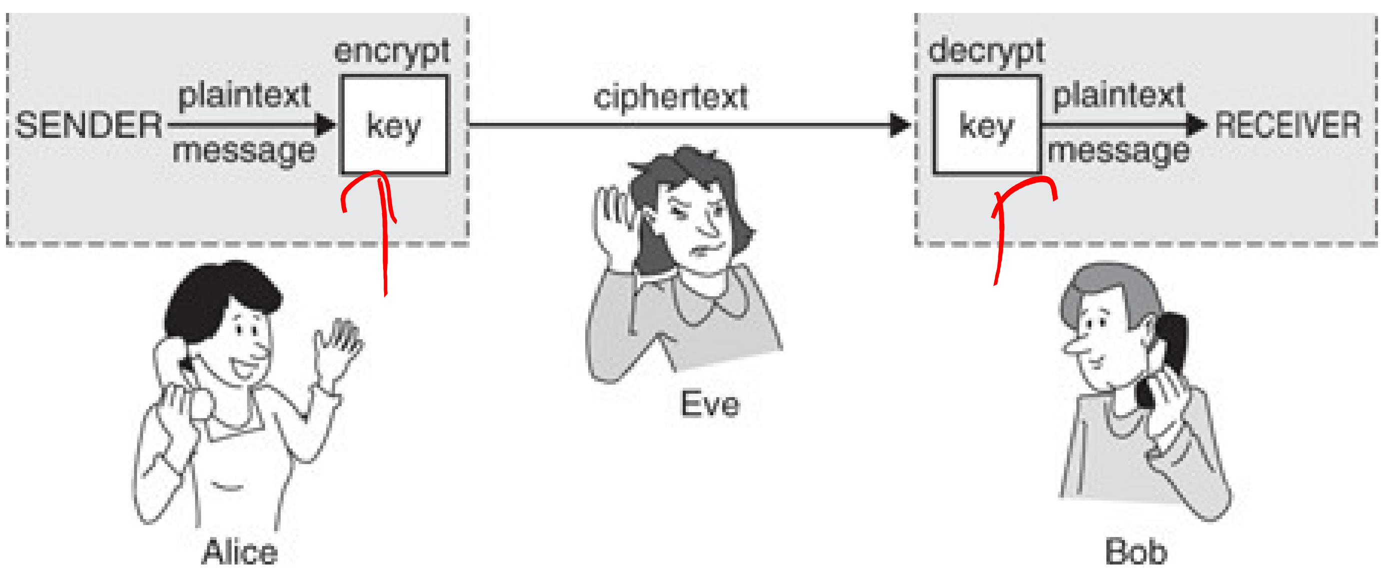
bitwise XOR

A Useful Identity

- If x and y are bits: $(x \oplus y) \oplus y = ?$ x
- What if x and y are bit-vectors?

Private Key Cryptography

- Alice wants to communicate message secretly to Bob so that eavesdropper Eve who hears their conversation cannot tell what Alice's message is.
- Alice and Bob can get together and privately share a secret key K ahead of time.



One-Time Pad

- Alice and Bob privately share random n-bit vector K
  - Eve does not know K
- Later, Alice has n-bit message m to send to Bob
  - Alice computes $C = m \oplus K$
  - Alice sends C to Bob
  - Bob computes $m = C \oplus K$ which is $(m \oplus K) \oplus K$ $= m$
- Eve cannot figure out m from C unless she can guess K