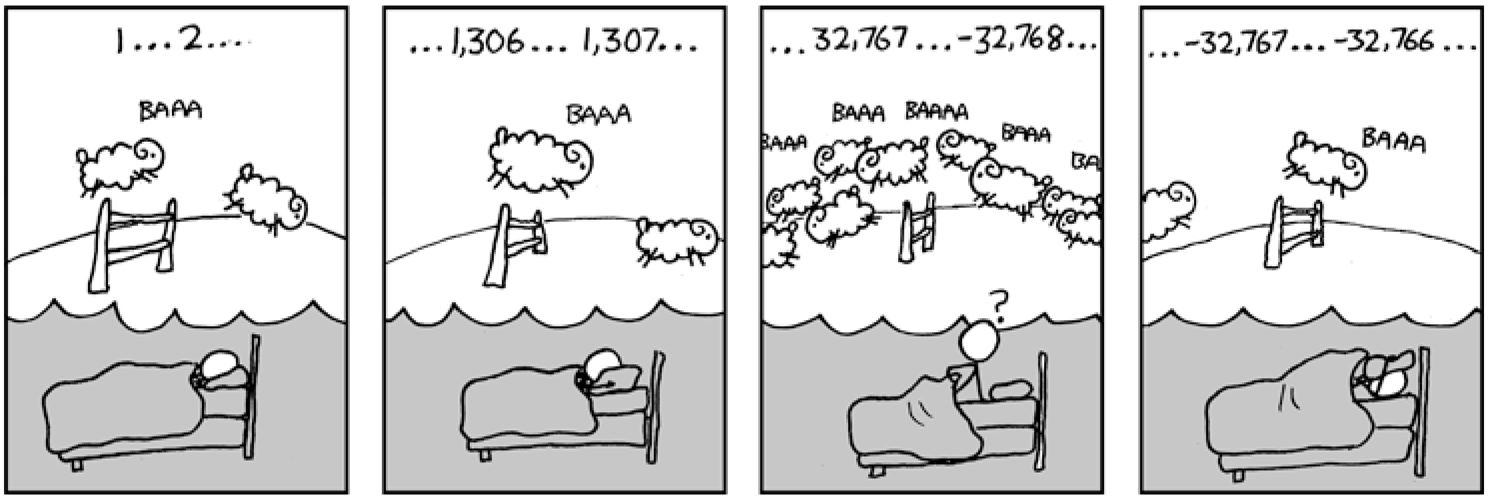
CSE 311: Foundations of Computing

Lecture 10: Modular Arithmetic



Last Class: Divisibility

Definition: "b divides a"

For a, b with $b \neq 0$:

$$b \mid a \leftrightarrow \exists q \ \ (a = qb)$$

Check Your Understanding. Which of the following are true?

$5 \mid 1$	$25 \mid 5$	$5 \mid 0$	$3 \mid 2$
5 | 1 iff 1 = 5k	25 | 5 iff 5 = 25k	5 | 0 iff 0 = 5k	3 | 2 iff 2 = 3k
$1 \mid 5$	$5 \mid 25$	$0 \mid 5$	$2 \mid 3$
1 | 5 iff 5 = 1k	5 | 25 iff 25 = 5k	0 | 5 iff 5 = 0k	2 | 3 iff 3 = 2k

Last class: Division Theorem

Domain of Discourse
Integers

Division Theorem

For a, b with $b > 0$
there exist *unique* integers q, r with $0 \leq r < b$
such that $a = qb + r$.

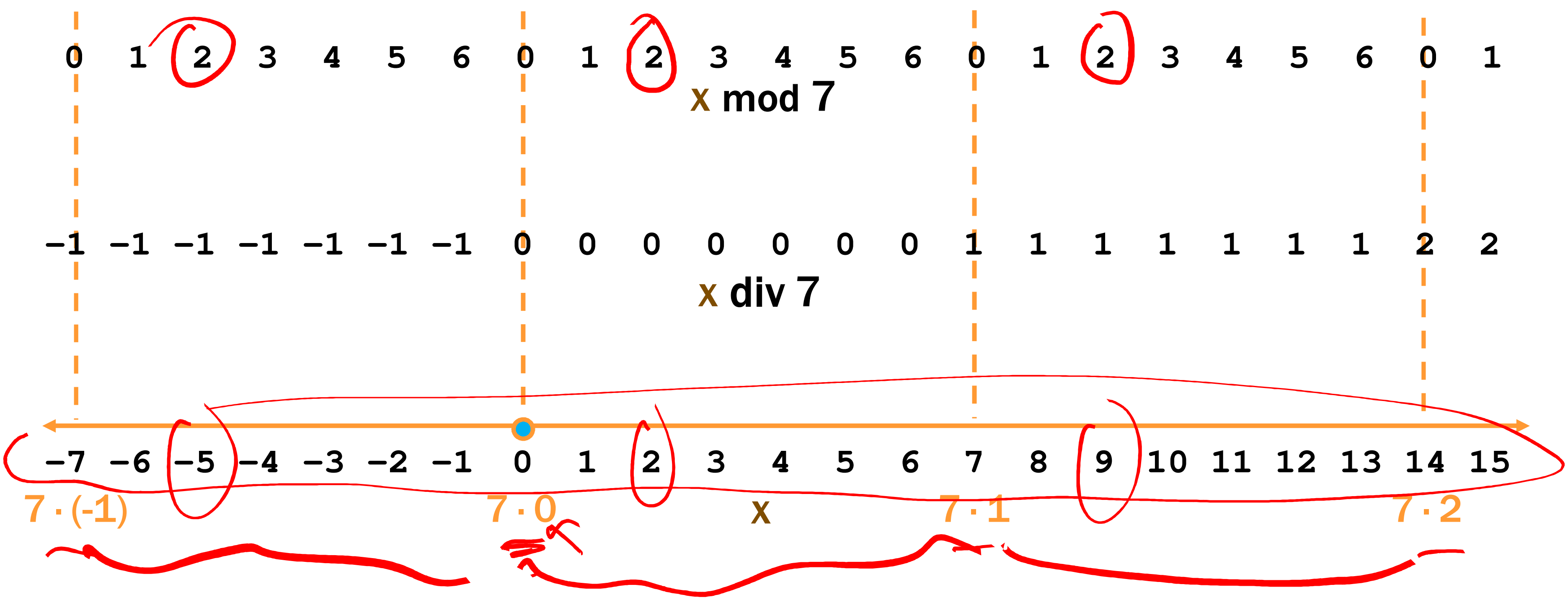
To put it another way, if we divide b **into** a**, we get a unique quotient** $\boxed{q = a \textbf{ div } b}$
and non-negative remainder $\boxed{r = a \textbf{ mod } b}$

$b = 2$

Note: r ≥ 0 even if a < 0.
Not quite the same as **a %** ~~**d**~~ b**.**

Last class: div and mod

$$\boxed{x = 7 \cdot (x \text{ div } 7) + (x \text{ mod } 7)}$$



Arithmetic, mod 7

(a + b) mod 7

(a × b) mod 7

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Modular Arithmetic

Domain of Discourse
Integers

(equivalent)

Definition: "a is congruent to b modulo m"

For a, b, m with $m > 0$

$$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$$

New notion of "sameness" or "equivalence" that will help us understand modular arithmetic.

This is a predicate (T/F values) on integers a, b, m. It does not produce numbers as output.

There is really a notion of sameness for each $m > 0$.
It may help you to think of $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ for a fixed $m > 0$ as an equivalence $\boldsymbol{a} \equiv_m \boldsymbol{b}$.
Standard math notation writes the $\pmod{m}$ on the right to tell you what notion of sameness $\equiv$ means.

Modular Arithmetic

Domain of Discourse
Integers

Definition: "a is congruent to b modulo m"

For a, b, m with $m > 0$

$$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$$

A chain of equivalences is written

$$a \equiv b \equiv c \equiv d \pmod{m}$$

This means $a \equiv b \pmod{m}$
and $b \equiv c \pmod{m}$
and $c \equiv d \pmod{m}$

Modular Arithmetic

Domain of Discourse
Integers

Definition: "a is congruent to b modulo m"

For a, b, m with $m > 0$

$$a \equiv b \ (\text{mod}\ m) \leftrightarrow m \mid (a - b)$$

$2 \mid (x-0)$

Check Your Understanding. What do each of these mean? When are they true?

True when x is even

x ≡ 0 (mod 2)

This statement is the same as saying "x is even"; so, any x that is even (including negative even numbers) will work.

$5 \mid (-1-19)$?

$5 \mid -20$

-1 ≡ 19 (mod 5)

This statement is true. 19 - (-1) = 20 which is divisible by 5

y ≡ 2 (mod 7)

$7 \mid (y-2)$ $\therefore 7k = y-2$ for some k

$y = 7k+2$

This statement is true for y in { ..., -12, -5, 2, 9, 16, ...}. In other words, all y of the form 2+7k for k an integer.

Modular Arithmetic: A Property

$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ be integers with $\boldsymbol{m > 0}$.
Then, $\boldsymbol{a \equiv b} \pmod{\boldsymbol{m}}$ if and only if $\boldsymbol{a \bmod m = b \bmod m}$.

Modular Arithmetic: A Property

$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$

Let $\boldsymbol{a, b, m}$ be integers with $\boldsymbol{m > 0}$.
Then, $\boldsymbol{a \equiv b \pmod{m}}$ if and only if $\boldsymbol{a \bmod m = b \bmod m}$.

($\Leftarrow$) Suppose that $a \bmod m = b \bmod m$.

By the division theorem, $a = mq + (a \bmod m)$ and
$b = ms + (b \bmod m)$ for some integers q,s.

Goal: show $a \equiv b \pmod{m}$, i.e., $m \mid (a - b)$.

Modular Arithmetic: A Property

$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$

Let $\boldsymbol{a, b, m}$ be integers with $\boldsymbol{m > 0}$.
Then, $\boldsymbol{a \equiv b \ (\mathrm{mod}\ m)}$ if and only if $\boldsymbol{a \bmod m = b \bmod m}$.

($\Leftarrow$) Suppose that $a \bmod m = b \bmod m$.

By the division theorem, $a = mq + (a \bmod m)$ and
$b = ms + (b \bmod m)$ for some integers q,s.

Then, $a - b = (mq + (\underline{a \bmod m})) - (ms + (\underline{b \bmod m}))$ $= mq - ms$
$= m(q - s) + (a \bmod m - b \bmod m)$ $= m(q-s)$
$= m(q - s)$ since $a \bmod m = b \bmod m$

Goal: show $a \equiv b \pmod{m}$, i.e., $m \mid (a - b)$.

Modular Arithmetic: A Property

$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$

Let $\boldsymbol{a, b, m}$ be integers with $\boldsymbol{m > 0}$.
Then, $\boldsymbol{a \equiv b \pmod{m}}$ if and only if $\boldsymbol{a \bmod m = b \bmod m}$.

(⇐) Suppose that $a \bmod m = b \bmod m$.

By the division theorem, $a = mq + (a \bmod m)$ and
$b = ms + (b \bmod m)$ for some integers q,s.

Then, $a - b = (mq + (a \bmod m)) - (ms + (b \bmod m))$
$= m(q - s) + (a \bmod m - b \bmod m)$
$= m(q - s)$ since $a \bmod m = b \bmod m$

Therefore, $m \mid (a - b)$ and so $a \equiv b \pmod{m}$.

Goal: show $a \equiv b \pmod{m}$, i.e., $m \mid (a - b)$.

(Halfway there)

Modular Arithmetic: A Property

$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$

Let $\boldsymbol{a, b, m}$ be integers with $\boldsymbol{m > 0}$.
Then, $\boldsymbol{a \equiv b \pmod{m}}$ if and only if $\boldsymbol{a \bmod m = b \bmod m}$.

(⇒) Suppose that $a \equiv b \pmod{m}$. $\therefore \ m \mid (a-b)$

Then, $m \mid (a - b)$ by definition of congruence.
So, $a - b = km$ for some integer k by definition of divides.
Therefore, $a = b + km$.

Goal: show $a \bmod m \equiv b \bmod m$

Modular Arithmetic: A Property

$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$

Let $\boldsymbol{a, b, m}$ be integers with $\boldsymbol{m > 0}$.
Then, $\boldsymbol{a \equiv b \pmod{m}}$ if and only if $\boldsymbol{a \bmod m = b \bmod m}$.

(⇒) Suppose that $a \equiv b \pmod{m}$.

Then, $m \mid (a - b)$ by definition of congruence.
So, $a - b = km$ for some integer k by definition of divides.
Therefore, $a = b + km$.

By the Division Theorem, we have $a = qm + (a \bmod m)$,
where $0 \leq (a \bmod m) < m$.

Goal: show $a \bmod m \equiv b \bmod m$

Modular Arithmetic: A Property

$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$

Let $\boldsymbol{a, b, m}$ be integers with $\boldsymbol{m > 0}$.
Then, $\boldsymbol{a \equiv b \pmod{m}}$ if and only if $\boldsymbol{a \bmod m = b \bmod m}$.

(⇒) Suppose that $a \equiv b \pmod{m}$.

Then, $m \mid (a - b)$ by definition of congruence.
So, $a - b = km$ for some integer k by definition of divides.
Therefore, $a = b + km$.

By the Division Theorem, we have $a = qm + (a \bmod m)$,
where $0 \leq (a \bmod m) < m$.

Combining these, we have $qm + (a \bmod m) = a = b + km$
or equiv., $b = qm - km + (a \bmod m) = (q - k)m + (a \bmod m)$.
By the Division Theorem, we have $b \bmod m = a \bmod m$. ■

Goal: show $a \bmod m \equiv b \bmod m$

Modular Arithmetic: A Property

$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$

Let $\boldsymbol{a, b, m}$ be integers with $\boldsymbol{m > 0}$.
Then, $\boldsymbol{a \equiv b} \pmod{\boldsymbol{m}}$ if and only if $\boldsymbol{a \bmod m = b \bmod m}$.

(⇒) Suppose that $a \equiv b \pmod{m}$.

Then, $m \mid (a - b)$ by definition of congruence.
So, $a - b = km$ for some integer k by definition of divides.
Therefore, $a = b + km$.

By the Division Theorem, we have $a = qm + (a \bmod m)$,
where $0 \leq (a \bmod m) < m$.

Combining these, we have $qm + (a \bmod m) = a = b + km$
or equiv., $b = qm - km + (a \bmod m) = (q - k)m + (a \bmod m)$.
By the Division Theorem, we have $b \bmod m = a \bmod m$. ∎

Goal: show $a \bmod m \equiv b \bmod m$

Modular Arithmetic: A Property

Let $\boldsymbol{a, b, m}$ be integers with $\boldsymbol{m} >$
Then, $\boldsymbol{a \equiv b \ (\mathrm{mod}\ m)}$ if and only

In future, we will usually go directly between these without discussing “divides” every time.

(⇒) Suppose that $a \equiv b\ (\mathrm{mod}\ m)$.

Then, $m \mid (a - b)$ by definition of congruence.
So, $a - b = km$ for some integer k by definition of divides.
Therefore, $a = b + km$.

By the Division Theorem, we have $a = qm + (a \bmod m)$,
where $0 \le (a \bmod m) < m$.

Combining these, we have $qm + (a \bmod m) = a = b + km$
or equiv., $b = qm - km + (a \bmod m) = (q - k)m + (a \bmod m)$.
By the Division Theorem, we have $b \bmod m = a \bmod m$. ∎

The $\mathbf{mod}\ \boldsymbol{m}$ function vs the $\equiv \pmod{\boldsymbol{m}}$ predicate

- What we have just shown
  - The $\mathbf{mod}\ \boldsymbol{m}$ function maps any integer $\boldsymbol{a}$ to a remainder $\boldsymbol{a}\ \mathbf{mod}\ \boldsymbol{m} \in \{0,1,..,\boldsymbol{m}-1\}$.
  - Imagine grouping together all integers that have the same value of the $\mathbf{mod}\ \boldsymbol{m}$ function

    That is, the same remainder in $\{0,1,..,\boldsymbol{m}-1\}$.
  - The $\equiv \pmod{\boldsymbol{m}}$ predicate compares integers $\boldsymbol{a}, \boldsymbol{b}$. It is true if and only if the $\mathbf{mod}\ \boldsymbol{m}$ function has the same value on $\boldsymbol{a}$ and on $\boldsymbol{b}$.

    That is, $\boldsymbol{a}$ and $\boldsymbol{b}$ are in the same group.

Recall: Familiar Properties of "="

- **If $a = b$ and $b = c$, then $a = c$.**
  - **i.e., if $a = b = c$, then $a = c$**

- **If $a = b$ and $c = d$, then $a + c = b + d$.**
  - **in particular, since $c = c$ is true, we can "$+ c$" to both sides**

- **If $a = b$ and $c = d$, then $ac = bd$.**
  - **in particular, since $c = c$ is true, we can "$\times c$" to both sides**

These are the facts that allow us to use algebra to solve problems

Modular Arithmetic: Basic Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{b} \equiv \boldsymbol{c} \pmod{m}$,
then $\boldsymbol{a} \equiv \boldsymbol{c} \pmod{m}$**.**

Modular Arithmetic: Basic Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{b} \equiv \boldsymbol{c} \pmod{m}$,
then $\boldsymbol{a} \equiv \boldsymbol{c} \pmod{m}$**.**

Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$.

Modular Arithmetic: Basic Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{b} \equiv \boldsymbol{c} \pmod{m}$,
then $\boldsymbol{a} \equiv \boldsymbol{c} \pmod{m}$.

Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$.
Then, by the previous property, we have
$a \bmod m = b \bmod m$ and $b \bmod m = c \bmod m$.

Putting these together, we have $a \bmod m = c \bmod m$, which says that $a \equiv c \pmod{m}$, by the previous property.

■

Modular Arithmetic: Addition Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{m}$
then $\boldsymbol{a} + \boldsymbol{c} \equiv \boldsymbol{b} + \boldsymbol{d} \pmod{m}$.

Modular Arithmetic: Addition Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{m}$
then $\boldsymbol{a} + \boldsymbol{c} \equiv \boldsymbol{b} + \boldsymbol{d} \pmod{m}$.

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

Modular Arithmetic: Addition Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{m}$
then $\boldsymbol{a} + \boldsymbol{c} \equiv \boldsymbol{b} + \boldsymbol{d} \pmod{m}$.

Suppose ~~that~~ $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
Unrolling the definitions, we can see that $a - b = km$ and $c - d = jm$ for some integers k, j.

$a - b + c - d$
$= km$
$+ jm$

Adding the equations together gives us
$(a + c) - (b + d) = m(k + j)$.

By the definition of congruence, we have $a + c \equiv b + d \pmod{m}$.

■

Modular Arithmetic: Multiplication Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{m}$
then $\boldsymbol{ac} \equiv \boldsymbol{bd} \pmod{m}$.

$a - b = km$ for some int k

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

$c - d = lm$ for some integer l

$\therefore$

Modular Arithmetic: Multiplication Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{m}$
then $\boldsymbol{ac} \equiv \boldsymbol{bd} \pmod{m}$.

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
Unrolling the definitions, we can see that $a - b = km$ and $c - d = jm$ for some integer k, j or equivalently, $a = km + b$ and $c = jm + d$.

Modular Arithmetic: Multiplication Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{m}$
then $\boldsymbol{ac} \equiv \boldsymbol{bd} \pmod{m}$.

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
Unrolling the definitions, we can see that $a - b = km$ and $c - d = jm$ for some integer k, j or equivalently, $a = km + b$ and $c = jm + d$.

Multiplying both together gives us $ac = (km + b)(jm + d) = \underline{kjm^2 + kmd + bjm} + bd$.

Modular Arithmetic: Multiplication Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{m}$
then $\boldsymbol{ac} \equiv \boldsymbol{bd} \pmod{m}$.

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
Unrolling the definitions, we can see that $a - b = km$ and $c - d = jm$ for some integer k, j or equivalently, $a = km + b$ and $c = jm + d$.

Multiplying both together gives us $ac = (km + b)(jm + d) = kjm^2 + kmd + bjm + bd$. Re-arranging, this becomes $ac - bd = m(kjm + kd + bj)$.

Modular Arithmetic: Multiplication Property

Let m be a positive integer.
If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{m}$
then $\boldsymbol{ac} \equiv \boldsymbol{bd} \pmod{m}$.

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
Unrolling the definitions, we can see that $a - b = km$ and $c - d = jm$ for some integer k, j or equivalently, $a = km + b$ and $c = jm + d$.

Multiplying both together gives us $ac = (km + b)(jm + d) = kjm^2 + kmd + bjm + bd$. Re-arranging, this becomes $ac - bd = m(kjm + kd + bj)$.

This says $ac \equiv bd \pmod{m}$ by the definition of congruence. ■

Modular Arithmetic: Properties

If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{b} \equiv \boldsymbol{c} \pmod{m}$ then $\boldsymbol{a} \equiv \boldsymbol{c} \pmod{m}$

If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{m}$ then

$$\boldsymbol{a} + \boldsymbol{c} \equiv \boldsymbol{b} + \boldsymbol{d} \pmod{m} \text{ and}$$

$$\boldsymbol{ac} \equiv \boldsymbol{bd} \pmod{m}$$

Corollary: If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{m}$ then

$$\boldsymbol{a} + \boldsymbol{c} \equiv \boldsymbol{b} + \boldsymbol{c} \pmod{m} \text{ and}$$

$$\boldsymbol{ac} \equiv \boldsymbol{bc} \pmod{m}$$

These allow us to solve problems in modular arithmetic, e.g.

- **add/subtract numbers from both sides of equations**
- **multiply numbers on both sides of equations.**
- **use chains of equivalences**

Example: Proof by Cases with mod

n even

Let $\boldsymbol{n}$ be an integer. Prove that $\boldsymbol{n^2 \equiv 0} \pmod 4$ or $\boldsymbol{n^2 \equiv 1} \pmod 4$.

n odd

Let's start by looking at small examples:

$\mathbf{0^2} = \mathbf{0} \equiv \mathbf{0} \pmod 4$

$\mathbf{1^2} = \mathbf{1} \equiv \mathbf{1} \pmod 4$

$\mathbf{2^2} = \mathbf{4} \equiv \mathbf{0} \pmod 4$

$\mathbf{3^2} = \mathbf{9} \equiv \mathbf{1} \pmod 4$

$\mathbf{4^2} = \mathbf{16} \equiv \mathbf{0} \pmod 4$

It looks as though we have:

If $\boldsymbol{n}$ is even then $\boldsymbol{n^2 \equiv 0} \pmod 4$

If $\boldsymbol{n}$ is odd then $\boldsymbol{n^2 \equiv 1} \pmod 4$

Example: Proof by Cases with mod

Let $\boldsymbol{n}$ be an integer. Prove that $\boldsymbol{n^2 \equiv 0} \pmod{4}$ or
$\boldsymbol{n^2 \equiv 1} \pmod{4}$.

Case 1 (n is even):

Suppose n is even.
Then, $n = 2k$ for some integer k.
So, $n^2 = (2k)^2 = 4k^2 = 4k^2 + 0$.
So, by the definition of congruence,
we have $n^2 \equiv 0 \pmod{4}$.

Example: Proof by Cases with mod

Let $\boldsymbol{n}$ be an integer. Prove that ($\boldsymbol{n^2 \equiv 0 \pmod 4}$ or $\boldsymbol{n^2 \equiv 1 \pmod 4}$.)

Case 1 (n is even): Done.

Case 2 (n is odd):

Suppose n is odd.

Then, $n = 2k + 1$ for some integer k.

So, $n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 4(k^2 + k) + 1$.

So, by definition of congruence,

we have $n^2 \equiv 1 \pmod 4$.

Result follows by proof by cases since n is either even or odd

$7y + 3x \equiv 4 \pmod 9$

$x \equiv 7 \pmod 9$

$7y + 21 \equiv 4 \pmod 9$

$3x \equiv 21 \pmod 9$

$7y \equiv 1 \pmod 9$

$7y + 3 \equiv 4 \pmod 9$