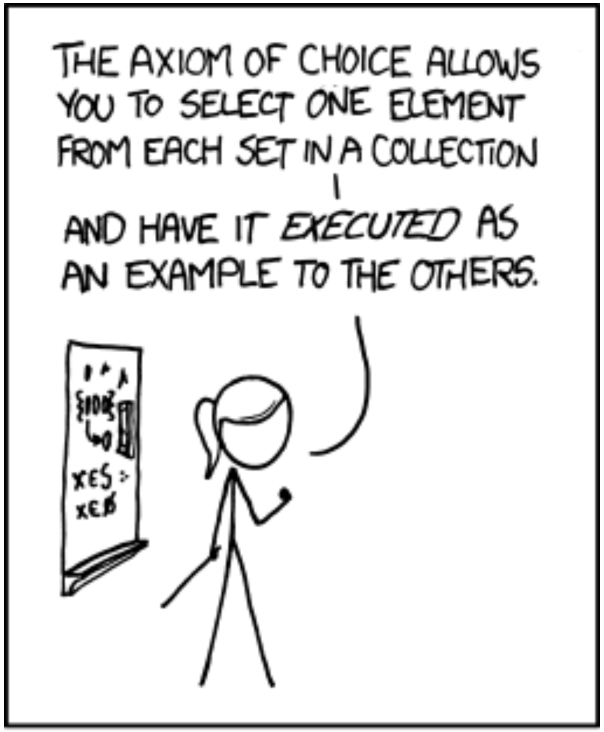
CSE 311: Foundations of Computing

Lecture 8: Predicate Logic Proofs, English Proofs



MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

Last class: Inference Rules for Quantifiers

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore \; P(a) \textbf{ for any } a}$$

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore P(c) \textbf{ for some } \textbf{\textit{special}}{**}\ c}$$

Intro ∀

** by special, we mean that c is a name for a value where P(c) is true. We can't use anything else about that value, so c has to be a NEW name!.

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) := $\exists$y (x = 2·y)
Odd(x) := $\exists$y (x = 2·y + 1)

Prove "There is an even number"

Formally: prove $\exists x\ \text{Even}(x)$

Elim

1. $0 = 2 \cdot 0$ algebra
2. $\exists y\,(0 = 2 \cdot y)$ intro $\exists$: 1
3. Even(0) def of Even: 2
4. $\exists x.\ \text{Even}(x)$ intro $\exists$: 3

Intro

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
$\mathrm{Even}(x) := \exists y\,(x = 2 \cdot y)$
$\mathrm{Odd}(x) := \exists y\,(x = 2 \cdot y + 1)$

Prove "There is an even number"

Formally: prove $\exists x\, \mathrm{Even}(x)$

1.	$\mathbf{2 = 2 \cdot 1}$	**Algebra**
2.	$\exists y\,(\mathbf{2} = \mathbf{2} \cdot y)$	**Intro** $\exists$**: 1**
3.	$\mathrm{Even}(\mathbf{2})$	**Definition of** Even**: 2**
4.	$\exists x\, \mathrm{Even}(x)$	**Intro** $\exists$**: 3**

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) := $\exists y\ (x = 2 \cdot y)$
Odd(x) := $\exists y\ (x = 2 \cdot y + 1)$
Prime(x) := "x > 1 and $x \neq a \cdot b$ for all integers a, b with $1<a<x$"

Prove "There is an even prime number"

Formally: prove $\exists x\ (\text{Even}(x) \land \text{Prime}(x))$

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) := ∃y (x = 2·y)
Odd(x) := ∃y (x = 2·y + 1)
Prime(x) := "x > 1 and x≠a·b for all integers a, b with 1<a<x"

Prove "There is an even prime number"

Formally: prove ∃x (Even(x) ∧ Prime(x))

1.	**2 = 2·1**	**Algebra**
2.	∃y (**2** = **2**·y)	**Intro** ∃: **1**
3.	Even(**2**)	**Def of** Even: **3**
4.	Prime(**2**)*	**Property of integers**
5.	Even(**2**) ∧ Prime(**2**)	**Intro** ∧: **2, 4**
6.	∃x (Even(x) ∧ Prime(x))	**Intro** ∃: **5**

*** Later we will further break down** "Prime" **using quantifiers to prove statements like this**

Inference Rules for Quantifiers: First look

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore \quad P(a) \quad \text{(for any a)}}$$

$A \not\Rightarrow B$

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some } \textit{special}^{**} \text{ c}}$$

Intro ∀

$$\frac{\text{“Let a be arbitrary*”}\ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

** c is a NEW name.

* in the domain of P

not like the others

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	Elim ∃
$\dfrac{\text{“Let a be arbitrary*”...}P(a)}{\therefore\ \forall x\, P(x)}$	$\dfrac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some special** } c}$

Prove: “The square of any even number is even.”

Formal proof of: $\forall x\, (\text{Even}(x) \rightarrow \text{Even}(x^2))$

3. $\forall x\, (\text{Even}(x) \rightarrow \text{Even}(x^2))$?

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	“Let a be arbitrary*” ... P(a) / ∴ ∀x P(x)	Elim ∃	∃x P(x) / ∴ P(c) for some *special*** c

Prove: “The square of any even number is even.”

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$?
3. $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$ Intro ∀: 1,2

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	Elim ∃
"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	∃x P(x) ∴ P(c) for some ***special***** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\mathrm{Even}(x) \rightarrow \mathrm{Even}(x^2))$

1. **Let a be an arbitrary integer**

elim

2.1 Even(**a**) Assumption

-intro

2.6 Even($\mathbf{a}^2$) ?

2. Even(**a**)→Even($\mathbf{a}^2$) Direct proof

3. ∀x (Even(x)→Even(x^2)) Intro ∀: 1,2

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Even and Odd

Intro ∀	"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	Elim ∃	∃x P(x) ∴ P(c) **for some *special*** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. **Let a be an arbitrary integer**

2.1	Even(**a**)	Assumption
2.2	$\exists y\,(\mathbf{a} = 2y)$	Definition of Even
2.5	$\exists y\,(\mathbf{a}^2 = 2y)$	?
2.6	Even($\mathbf{a}^2$)	Definition of Even

2. Even(**a**)→Even($\mathbf{a}^2$) Direct Proof
3. $\forall x\,(\text{Even}(x)\rightarrow\text{Even}(x^2))$ Intro ∀: 1,2

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Even and Odd

Intro ∀	"Let a be arbitrary*"...P(a) / ∴ ∀x P(x)	Elim ∃	∃x P(x) / ∴ P(c) for some *special*** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\, (\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. **Let a be an arbitrary integer**

2.1	Even(**a**)	Assumption
2.2	∃y (**a** = 2y)	Definition of Even
2.5	∃y (**a**^{2} = 2y)	Intro ∃: ?
2.6	Even(**a**^{2})	Definition of Even

2. Even(**a**)→Even(**a**^{2}) — Direct proof
3. $\forall x\, (\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

Need **a**^{2} = 2c for some **c**

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Even and Odd

Intro ∀

$$\frac{\text{“Let } a \text{ be arbitrary*”} \ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

1. Let **a** be an arbitrary integer
 2.1 Even(**a**) — Assumption
 2.2 ∃y (**a** = 2y) — Definition of Even
 2.3 **a** = 2**b** — Elim ∃: **b**

 2.5 ∃y ($\mathbf{a}^2$ = 2y) — Intro ∃:
 2.6 Even($\mathbf{a}^2$) — Definition of Even
2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof
3. ∀x (Even(x)→Even(x^2)) — Intro ∀: 1,2

Prove: “The square of any even number is even.”

Formal proof of: ∀x (Even(x) → Even(x^2))

$a^2 = (2b)^2$
$= 4b^2$
$= 2(2b^2)$

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	∃y (**a** = 2y)	Definition of Even
2.3	**a** = 2**b**	Elim ∃: **b**
2.5	∃y ($\mathbf{a}^2$ = 2y)	Intro ∃: ?
2.6	Even($\mathbf{a}^2$)	Definition of Even

2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof
3. ∀x (Even(x)→Even(x^2)) — Intro ∀: 1,2

Need $\mathbf{a}^2$ = 2c for some c

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	Elim ∃
"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	∃x P(x) ∴ P(c) for some **special**** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	$\exists y\,(\mathbf{a} = 2y)$	Definition of Even
2.3	$\mathbf{a} = 2\mathbf{b}$	Elim ∃: **b**
2.4	$\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$	Algebra
2.5	$\exists y\,(\mathbf{a}^2 = 2y)$	Intro ∃
2.6	Even($\mathbf{a}^2$)	Definition of Even

Used $\mathbf{a}^2 = 2c$ for $c=2\mathbf{b}^2$

2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$ Direct Proof
3. $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$ Intro ∀: 1,2

These rules need some caveats…

There are extra conditions on using these rules:

Intro ∀

$$\frac{\text{“Let } a \text{ be arbitrary*”} \ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

* in the domain of P. No other name in P depends on a

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some } \textit{special}^{**}\ c}$$

** c is a NEW name.
List all dependencies for c.

Without those rules, it is possible to infer claims that are false

Without the rules, one could infer false claims...

There are extra conditions on using these rules:

Intro $\forall$

$$\frac{\text{“Let } a \text{ be arbitrary*”} \ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

* in the domain of P

Elim $\exists$

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some } \textit{special}^{**}\ c}$$

** c has to be a NEW name.

Over integer domain: $\forall x\, \exists y\, (y \neq x)$ **is True but** $\exists y \forall x\, (y \neq x)$ **is False**

BAD “PROOF”

1. $\forall x\, \exists y\, (y \neq x)$ — Given
2. **Let a be an arbitrary integer**
3. $\exists y\, (y \neq a)$ — Elim $\forall$: 1
4. $b \neq a$ — Elim $\exists$: 3 (b new constant)
5. $\forall x\, (b \neq x)$ — Intro $\forall$: 2,4
6. $\exists y \forall x\, (y \neq x)$ — Intro $\exists$: 5

With the extra conditions we can kill the bad proof...

There are extra conditions on using these rules:

Intro $\forall$ — "Let a be arbitrary*"...P(a) ∴ $\forall x\, P(x)$

* in the domain of P. No other name in P depends on a

Elim $\exists$ — $\exists x\, P(x)$ ∴ P(c) for some *special*** c

** c is a NEW name.
List all dependencies for c.

Over integer domain: $\forall x\, \exists y\, (y \neq x)$ **is True but** $\exists y \forall x\, (y \neq x)$ **is False**

BAD "PROOF" KILLED

1. $\forall x\, \exists y\, (y \neq x)$ Given
2. **Let a be an arbitrary integer**
3. $\exists y\, (y \neq a)$ Elim $\forall$: 1
4. $b \neq a$ Elim $\exists$: 3 (b depends on a)
5. ~~$\forall x\, (b \neq x)$ Intro $\forall$: 2,4~~
6. $\exists y \forall x\, (y \neq x)$ Intro $\exists$: 5

Can't get rid of a since another name in the same line, b, depends on it!

Inference Rules for Quantifiers: Full version

Intro $\exists$

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim $\forall$

$$\frac{\forall x\, P(x)}{\therefore \quad P(a) \quad \textbf{(for any } a)}$$

Elim $\exists$

$$\frac{\exists x\, P(x)}{\therefore P(c) \textbf{ for some } \textbf{\textit{special}}^{**}\ c}$$

Intro $\forall$

$$\frac{\text{“}\textbf{Let } a \textbf{ be arbitrary}^*\text{”}\ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

** c is a NEW name.
List all dependencies for c.

* in the domain of P. No other name in P depends on a.

Formal Proofs

- In principle, formal proofs are the standard for what it means to be "proven" in mathematics
  - almost all math (and theory CS) done in Predicate Logic

- But they are tedious and impractical
  - e.g., applications of commutativity and associativity
  - Russell & Whitehead's formal proof that 1+1 = 2 *appears after more than 100 pages of build up*
  - we allowed ourselves to cite "Arithmetic", "Algebra", etc.

- Similar situation exists in programming…

Programming

```
a := ADD(i, 1)
b := MOD(a, n)
c := ADD(arr, b)
d := LOAD(c)
e := ADD(arr, i)
STORE(e, d)
```

```
arr[i] = arr[(i+1) % n];
```

Assembly Language

High-level Language

Programming vs Proofs

```
a := ADD(i, 1)
b := MOD(a, n)
c := ADD(arr, b)
d := LOAD(c)
e := ADD(arr, i)
STORE(e, d)
```

Assembly Language for Programs

Given

Given

Elim $\wedge$: 1

Double Negation: 4

Elim $\vee$: 3, 5

Modus Ponens: 2, 6

Assembly Language for Proofs

Proofs

Given

Given

$\wedge$ Elim: 1

Double Negation: 4

$\vee$ Elim: 3, 5

MP: 2, 6

what is the "Java" for proofs?

Assembly Language for Proofs

High-level Language for Proofs

Proofs

Given

Given

∧ Elim: 1

Double Negation: 4

∨ Elim: 3, 5

MP: 2, 6

English?

Assembly Language for Proofs

High-level Language for Proofs

Proofs

Given

Given

$\wedge$ Elim: 1

Double Negation: 4

$\vee$ Elim: 3, 5

MP: 2, 6

Math English

Assembly Language for Proofs

High-level Language for Proofs

Proofs

- Formal proofs follow simple well-defined rules and should be easy for a machine to check
  - as assembly language is easy for a machine to execute

- English proofs correspond to those rules but are designed to be easier for humans to read
  - also easy to check with practice

    (almost all actual math and theory CS is done this way)
  - English proof is correct if the <u>reader</u> is convinced that they could translate it into a formal proof

    (the reader is the “compiler” for English proofs)

Formal Proof: Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove: "The square of every even number is even."

Formal proof of: $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. **Let $\mathbf{a}$ be an arbitrary integer**

2.1	$\text{Even}(\mathbf{a})$	Assumption
2.2	$\exists y\ (\mathbf{a} = 2y)$	Definition of Even
2.3	$\mathbf{a} = 2\mathbf{b}$	Elim $\exists$
2.4	$\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$	Algebra
2.5	$\exists y\ (\mathbf{a}^2 = 2y)$	Intro $\exists$
2.6	$\text{Even}(\mathbf{a}^2)$	Definition of Even

2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$ — Direct Proof
3. $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro $\forall$

English Proof: Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Let a be an arbitrary integer.	1.	Let a be an arbitrary integer	
Suppose a is even.	2.1	Even(a)	Assumption
Then, by definition, $a = 2b$ for some integer b.	2.2	$\exists y\ (a = 2y)$	Definition
	2.3	$a = 2b$	Elim $\exists$
Squaring both sides, we get $a^2 = 4b^2 = \underbrace{2(2b^2)}$.	2.4	$a^2 = 4b^2 = 2(2b^2)$	Algebra
So a^2 is, by definition, even.	2.5	$\exists y\ (a^2 = 2y)$	Intro $\exists$
	2.6	Even(a^2)	Definition
Since a was arbitrary, we have shown that the square of every even number is even.	2.	Even(a)$\rightarrow$Even(a^2)	Direct Proof
	3.	$\forall x$ (Even(x)$\rightarrow$Even(x^2))	Intro $\forall$

English Proof: Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Proof: Let **a** be an arbitrary integer.

Suppose **a** is even. Then, by definition, $a = 2b$ for some integer **b**. Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$. So a^2 is, by definition, ~~is~~ even.

Since **a** was arbitrary, we have shown that the square of every even number is even. ■

English Proof: Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Proof: Let **a** be an arbitrary **even** integer.

Then, by definition, **a** = 2**b** for some integer b. Squaring both sides, we get $\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$. So $\mathbf{a}^2$ is, by definition, is even.

Since **a** was arbitrary, we have shown that the square of every even number is even. ∎

$$\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$$

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Even}(x+y))$

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\mathrm{Odd}(x) \land \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$

Let x and y be arbitrary integers.

1. **Let x be an arbitrary integer**
2. **Let y be an arbitrary integer**

Since x and y were arbitrary, the sum of any odd integers is even.

3. $(\mathrm{Odd}(\mathbf{x}) \land \mathrm{Odd}(\mathbf{y})) \rightarrow \mathrm{Even}(\mathbf{x}+\mathbf{y})$
4. $\forall \mathbf{y}\ ((\mathrm{Odd}(\mathbf{x}) \land \mathrm{Odd}(\mathbf{y})) \rightarrow \mathrm{Even}(\mathbf{x}+\mathbf{y}))$ Intro $\forall$
5. $\forall \mathbf{x} \forall \mathbf{y}\ ((\mathrm{Odd}(\mathbf{x}) \land \mathrm{Odd}(\mathbf{y})) \rightarrow \mathrm{Even}(\mathbf{x}+\mathbf{y}))$ Intro $\forall$

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Even}(x+y))$

Let x and y be arbitrary integers.

1. **Let x be an arbitrary integer**
2. **Let y be an arbitrary integer**

Suppose that both are odd.

3.1 $\text{Odd}(x) \land \text{Odd}(y)$ Assumption

so x+y is even.

3.9 $\text{Even}(x+y)$

Since x and y were arbitrary, the sum of any odd integers is even.

3. $(\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Even}(x+y)$ DPR
4. $\forall y\ ((\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Even}(x+y))$ Intro $\forall$
5. $\forall x \forall y\ ((\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Even}(x+y))$ Intro $\forall$

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\mathrm{Odd}(x) \wedge \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$

Let x and y be arbitrary integers.

1. Let **x** be an arbitrary integer
2. Let **y** be an arbitrary integer

Suppose that both are odd.

3.1	Odd(**x**) ∧ Odd(**y**)	Assumption
3.2	Odd(**x**)	Elim ∧: 2.1
3.3	Odd(**y**)	Elim ∧: 2.1

so x+y is even.

3.9	Even(**x+y**)	

Since x and y were arbitrary, the sum of any odd integers is even.

3.	(Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**)	DPR
4.	∀**y** ((Odd(**x**) ∧ Odd(**y**)) → Even(x+y))	Intro ∀
5.	∀**x**∀**y** ((Odd(**x**) ∧ Odd(**y**)) → Even(x+y))	Intro ∀

English Proof: Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Prove "The sum of two odd numbers is even."

Let x and y be arbitrary integers.

1. **Let x be an arbitrary integer**
2. **Let y be an arbitrary integer**

Suppose that both are odd.

3.1	Odd(**x**) ∧ Odd(**y**)	Assumption
3.2	Odd(**x**)	Elim ∧: 2.1
3.3	Odd(**y**)	Elim ∧: 2.1

Then, we have x = 2a+1 for some integer a and y = 2b+1 for some integer b.

3.4	∃**z** (**x** = 2**z**+1)	Def of Odd: 2.2
3.5	**x** = 2**a**+1	Elim ∃: 2.4
3.6	∃**z** (**y** = 2**z**+1)	Def of Odd: 2.3
3.7	**y** = 2**b**+1	Elim ∃: 2.5

so x+y is, by definition, even.

3.9	∃z (**x+y** = 2z)	Intro ∃: 2.4
3.10	Even(**x+y**)	Def of Even

Since x and y were arbitrary, the sum of any odd integers is even.

3.	(Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**)	DPR
4.	∀**y** ((Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**))	Intro ∀
5.	∀**x**∀**y** ((Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**))	Intro ∀

English Proof: Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The sum of two odd numbers is even."

$x+y = 2a+1 + 2b+1$
$= 2a+2b+2$
$= 2(a+b+1)$

English proof	Formal proof	Justification
Let x and y be arbitrary integers.	1. Let **x** be an arbitrary integer	
	2. Let **y** be an arbitrary integer	
	3.1 $\text{Odd}(\mathbf{x}) \land \text{Odd}(\mathbf{y})$	Assumption
Suppose that both are odd.	3.2 $\text{Odd}(\mathbf{x})$	Elim $\land$: 2.1
	3.3 $\text{Odd}(\mathbf{y})$	Elim $\land$: 2.1
Then, we have x = 2a+1 for some integer a and y = 2b+1 for some integer b.	3.4 $\exists \mathbf{z}\ (\mathbf{x} = 2\mathbf{z}+1)$	Def of Odd: 2.2
	3.5 $\mathbf{x} = 2\mathbf{a}+1$	Elim $\exists$: 2.4
	3.6 $\exists \mathbf{z}\ (\mathbf{y} = 2\mathbf{z}+1)$	Def of Odd: 2.3
	3.7 $\mathbf{y} = 2\mathbf{b}+1$	Elim $\exists$: 2.5
Their sum is x+y = ... = 2(a+b+1)	3.8 $\mathbf{x+y} = 2(\mathbf{a+b+1})$	Algebra
so x+y is, by definition, even.	3.9 $\exists z\ (\mathbf{x+y} = 2z)$	Intro $\exists$: 2.4
	3.10 $\text{Even}(\mathbf{x+y})$	Def of Even
Since x and y were arbitrary, the sum of any odd integers is even.	3. $(\text{Odd}(\mathbf{x}) \land \text{Odd}(\mathbf{y})) \rightarrow \text{Even}(\mathbf{x+y})$	DPR
	4. $\forall \mathbf{y}\ ((\text{Odd}(\mathbf{x}) \land \text{Odd}(\mathbf{y})) \rightarrow \text{Even}(\mathbf{x+y}))$	Intro $\forall$
	5. $\forall \mathbf{x} \forall \mathbf{y}\ ((\text{Odd}(\mathbf{x}) \land \text{Odd}(\mathbf{y})) \rightarrow \text{Even}(\mathbf{x+y}))$	Intro $\forall$

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

$\forall x \forall y.\ \text{Odd}(x) \wedge \text{Odd}(y) \rightarrow \text{Even}(x+y)$

Prove "The sum of two odd numbers is even."

Proof: Let x and y be arbitrary integers.

Suppose that both are odd. Then, we have x = 2a+1 for some integer a and y = 2b+1 for some integer b. Their sum is x+y = (2a+1) + (2b+1) = 2a+2b+2 = 2(a+b+1), so x+y is, by definition, even.

Since x and y were arbitrary, the sum of any two odd integers is even. ■

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■

$\forall x\ \forall y\ ((\mathrm{Odd}(x) \land \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$