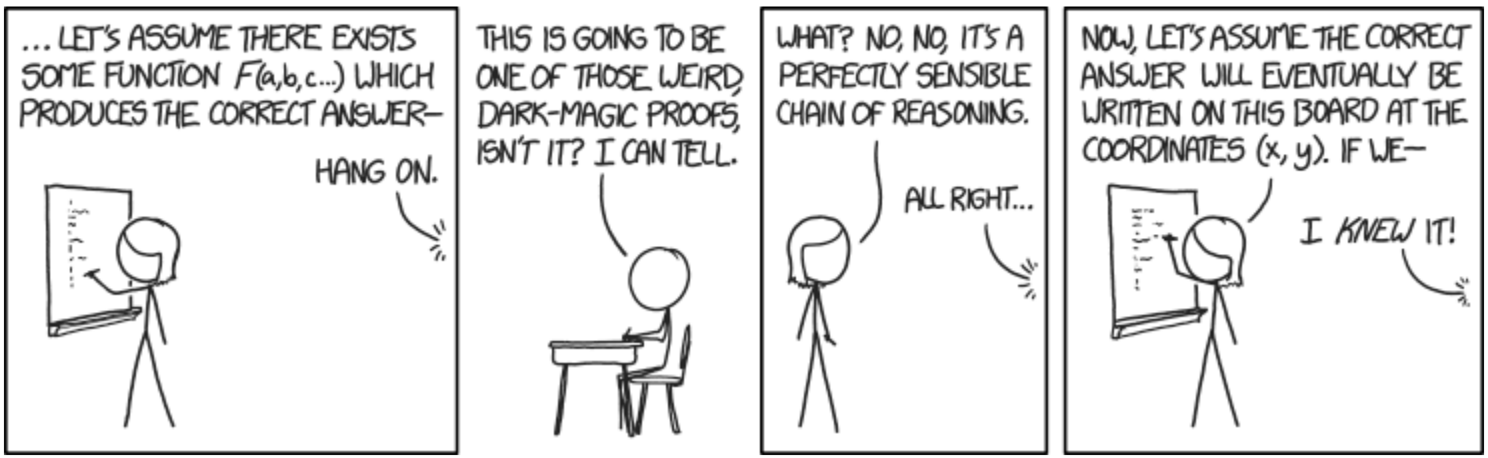
CSE 311: Foundations of Computing

Lecture 7: Propositional & Predicate Logic Proofs



- 2 HW corrections
- can resubmit before deadline

Last class: My First Proof!

Show that r follows from p, $p \to q$, and $q \to r$

1.	p	Given
2.	$p \to q$	Given
3.	$q \to r$	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

Modus Ponens $\dfrac{A\ ;\ A \to B}{\therefore\ B}$

Last class: Proofs can use equivalences too

Show that $\neg p$ follows from $p \to q$ and $\neg q$

1.	$p \to q$	Given
2.	$\neg q$	Given
3.	$\neg q \to \neg p$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Modus Ponens

1. $p \to q$ Given
2. $\neg q$ Given
3. $\neg q \to \neg p$ Contrapositive: 1
4. $\neg p$ MP: 2, 3

Last class: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim

Elim ∧: $$\frac{A \wedge B}{\therefore A, B}$$

Intro ∧: $$\frac{A \,;\, B}{\therefore A \wedge B}$$

Intro

Elim ∨: $$\frac{A \vee B \,;\, \neg A}{\therefore B}$$

Intro ∨: $$\frac{A}{\therefore A \vee B, B \vee A}$$

Elim →

Modus Ponens: $$\frac{A \,;\, A \rightarrow B}{\therefore B}$$

Intro →

Direct Proof: $$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Proofs

Show that r follows from p, $p \rightarrow q$ and $(p \wedge q) \rightarrow r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{A \; ; \; A \rightarrow B}{\therefore \; B}$$

$$\frac{A \wedge B}{\therefore \; A, B}$$

$$\frac{A \; ; \; B}{\therefore \; A \wedge B}$$

Proofs

Show that r follows from p, $p \rightarrow q$ and $(p \wedge q) \rightarrow r$

1. p Given
2. $p \rightarrow q$ Given
3. $(p \wedge q) \rightarrow r$ Given

9. r ??

$$\frac{A \; ; \; A \rightarrow B}{\therefore \; B}$$

$$\frac{A \wedge B}{\therefore \; A, B}$$

$$\frac{A \; ; \; B}{\therefore \; A \wedge B}$$

Proofs

Show that r follows from $p, p \to q$, and $(p \wedge q) \to r$

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!

1.	p	Given
2.	$p \to q$	Given
3.	q	MP: 1, 2
4.	$p \wedge q$	Intro $\wedge$: 1, 3
5.	$(p \wedge q) \to r$	Given
6.	r	MP: 4, 5

$$\dfrac{\dfrac{p \quad ; \quad \dfrac{p \quad ; \quad p \to q}{q}\,\text{MP}}{p \wedge q}\,\text{Intro } \wedge \quad ; \quad (p \wedge q) \to r}{r}\,\text{MP}$$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
20.	$\neg r$	?

First: Write down givens and goal

Idea: Work backwards!

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
20.	$\neg r$	MP: 2, ?

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 2 and 20 looks like "elim →" which is MP.

Proofs

Prove that $\neg r$ follows from $p \land s$, $q \to \neg r$, and $\neg s \lor q$.

1. $p \land s$ Given
2. $q \to \neg r$ Given
3. $\neg s \lor q$ Given

19. q ?

20. $\neg r$ MP: 2, 19

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- Now, we have a new "hole"
- We need to prove q...
  - Notice that at this point, if we prove q, we've proven $\neg r$...

Proofs

Prove that ¬r follows from p ∧ s, q → ¬r, and ¬s ∨ q.

1.	$p \land s$	Given
2.	$q \to \neg r$	Given
3.	$\neg s \lor q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

This looks like or-elimination.

Elim ∨ $\dfrac{A \lor B\ ;\ \neg A}{\therefore B}$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
18.	$\neg\neg s$	?
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

$\neg\neg s$ doesn't show up in the givens but s does and we can use equivalences

Proofs

Prove that $\neg r$ follows from $p \land s$, $q \rightarrow \neg r$, and $\neg s \lor q$.

1.	$p \land s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \lor q$	Given
17.	s	
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\lor$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

1. $p \land s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \lor q$ Given
17. s
18. $\neg\neg s$ Double Negation: 17
19. q $\lor$ Elim: 3, 18
20. $\neg r$ MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	$\wedge$ Elim: 1
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

No holes left! We just need to clean up a bit.

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
4.	s	$\wedge$ Elim: 1
5.	$\neg\neg s$	Double Negation: 4
6.	q	$\vee$ Elim: 3, 5
7.	$\neg r$	MP: 2, 6

Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

  e.g. $(p \rightarrow r) \vee q \equiv (\neg p \vee r) \vee q$

- Inference rules only can be applied to whole formulas (not correct otherwise).

  e.g. 1. $p \rightarrow r$ given

  ~~2. $(p \vee q) \rightarrow r$ intro $\vee$ from 1.~~

  Does not follow! e.g . p=F, q=T, r=F

Last class: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \wedge B}{\therefore A, B}$$

Intro ∧

$$\frac{A \;;\; B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B \;;\; \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Modus Ponens

$$\frac{A \;;\; A \rightarrow B}{\therefore B}$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules

Last class: New Perspective

Rather than comparing **A** and **B** as columns, zooming in on just the rows where A is true:

p	*q*	**A**	**B**
T	T	T	T
T	F	T	T
F	T	F	
F	F	F	

Given that A is true, we see that B is also true.

$A \Rightarrow B$

Last class: New Perspective

Rather than comparing **A** and **B** as columns, zooming in on just the rows where B is true:

p	q	A	B	$A \to B$
T	T	T	T	T
T	F	T	T	T
F	T	F	T	T
F	F	F	F	T

When we zoom out, what have we proven?

$$(A \to B) \equiv \mathbf{T}$$

To Prove An Implication: $A \rightarrow B$

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is **a proof** that "**Given** A, we can prove B."
- The direct proof rule:

  If you have such a proof then you can conclude that $A \rightarrow B$ is true

Proofs using the direct proof rule

Show that $p \to r$ follows from q and $(p \wedge q) \to r$

1. q Given
2. $(p \wedge q) \to r$ Given

This is a proof of $p \to r$

3.1. p Assumption

3.2.

3.3. r ??

If we know p is true... Then, we've shown r is true

3. $p \to r$ Direct Proof

Proofs using the direct proof rule

Show that $p \rightarrow r$ follows from q and $(p \wedge q) \rightarrow r$

1.	q	Given
2.	$(p \wedge q) \rightarrow r$	Given
3.1.	p	Assumption
3.2.	$p \wedge q$	Intro $\wedge$: 1, 3.1
3.3.	r	MP: 2, 3.2
3.	$p \rightarrow r$	Direct Proof

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens…

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

1.1.	$p \wedge q$	Assumption
1.2	p	Elim ∧ 1.1
1.3	q	Elim ∧ 1.1
1.9.	$p \vee q$	?? Intro ∨ ; 1.2
1.	$(p \wedge q) \rightarrow (p \vee q)$	Direct Proof

Example

Prove: $(p \land q) \to (p \lor q)$

1.1.	$p \land q$	Assumption
1.2.	p	Elim $\land$: 1.1
1.3.	$p \lor q$	Intro $\lor$: 1.2
1.	$(p \land q) \to (p \lor q)$	Direct Proof

One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example

Prove: $((p \to q) \wedge (q \to r)) \to (p \to r)$

1.1. $(p \to q) \wedge (q \to r)$ **Assumption**

1.? $p \to r$

1. $\left((p \to q) \wedge (q \to r)\right) \to (p \to r)$ **Direct Proof**

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

1.1.	$(p \rightarrow q) \wedge (q \rightarrow r)$	Assumption
1.2.	$p \rightarrow q$	$\wedge$ Elim: 1.1
1.3.	$q \rightarrow r$	$\wedge$ Elim: 1.1
1.?	$p \rightarrow r$	

1. $\big((p \rightarrow q) \wedge (q \rightarrow r)\big) \rightarrow (p \rightarrow r)$ Direct Proof

Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1. $(p \to q) \land (q \to r)$ Assumption

1.2. $p \to q$ $\land$ Elim: 1.1

1.3. $q \to r$ $\land$ Elim: 1.1

1.4.1. p Assumption

1.4.2 q MP: 1.2 1.4.1

1.4.? r MP: 1.3 1.4.2

1.4. $p \to r$ Direct Proof

1. $((p \to q) \land (q \to r)) \to (p \to r)$ Direct Proof

Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1.	$(p \to q) \land (q \to r)$	Assumption
1.2.	$p \to q$	$\land$ Elim: 1.1
1.3.	$q \to r$	$\land$ Elim: 1.1
1.4.1.	p	Assumption
1.4.2.	q	MP: 1.2, 1.4.1
1.4.3.	r	MP: 1.3, 1.4.2
1.4.	$p \to r$	Direct Proof
1.	$((p \to q) \land (q \to r)) \to (p \to r)$	Direct Proof

Inference Rules for Quantifiers: First look

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore\ P(a)}$$ **(for any** a)

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c)}$$ **for some *special*** ** c

Intro ∀

** By special, we mean that c is a name for a value where P(c) is true. We can't use anything else about that value, so c has to be a NEW name!

My First Predicate Logic Proof

Domain of Discourse
Integers

Prove $(\forall x\, P(x)) \rightarrow (\exists x\, P(x))$

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

5. $(\forall x\, P(x)) \rightarrow (\exists x\, P(x))$?

The main connective is implication so Direct Proof seems good

My First Predicate Logic Proof

Domain of Discourse
Integers

Intro ∃	P(c) for some c	
	∴ ∃x P(x)	
Elim ∀	∀x P(x)	
	∴ P(a) for any a	

Prove $\forall x\, P(x)) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ **Assumption**

1.2 P(5) ∀ Elim 1.1

We need an ∃ we don't have so "intro ∃" rule makes sense

1.5. $\exists x\, P(x)$?

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ **Direct Proof**

My First Predicate Logic Proof

Domain of Discourse
Integers

Intro ∃: P(c) for some c ∴ $\exists x\, P(x)$

Elim ∀: $\forall x\, P(x)$ ∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

We need an ∃ we don't have so "intro ∃" rule makes sense

1.5. $\exists x\, P(x)$ Intro ∃: ?

That requires P(c) for some c.

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof

My First Predicate Logic Proof

Domain of Discourse
Integers

Prove $\forall x\, P(x) \to \exists x\, P(x)$

Intro ∃: $\dfrac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$

1. $\forall x\, P(x) \to \exists x\, P(x)$ Direct Proof

1.1.	$\forall x\, P(x)$	Assumption
1.4.	$P(5)$	?
1.5.	$\exists x\, P(x)$	Intro ∃: 1.4
1.	$\forall x\, P(x) \to \exists x\, P(x)$	Direct Proof

My First Predicate Logic Proof

Domain of Discourse
Integers

Prove $\forall x\ P(x) \rightarrow \exists x\ P(x)$

Intro $\exists$: $\dfrac{P(c) \text{ for some } c}{\therefore\ \exists x\ P(x)}$

1. $\forall x\ P(x) \rightarrow \exists x\ P(x)$ Direct Proof

1.1.	$\forall x\ P(x)$	Assumption
1.4.	$P(5)$	Elim $\forall$: 1.1
1.5.	$\exists x\ P(x)$	Intro $\exists$: 1.4
1.	$\forall x\ P(x) \rightarrow \exists x\ P(x)$	Direct Proof

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ $\exists x\, P(x)$

Elim ∀: $\forall x\, P(x)$ ∴ P(a) for any a

Prove $\forall x\, P(x) \to \exists x\, P(x)$

1.1.	$\forall x\, P(x)$	Assumption
1.2.	$P(5)$	Elim ∀: 1.1
1.3.	$\exists x\, P(x)$	Intro ∃: 1.2
1.	$\forall x\, P(x) \to \exists x\, P(x)$	Direct Proof

Working forwards as well as backwards:
In applying "Intro ∃" rule we didn't know what expression we might be able to prove P(c) for, so we worked forwards to figure out what might work.

Predicate Logic Proofs

- Can use
  - Predicate logic inference rules

    whole formulas only
  - Predicate logic equivalences (De Morgan's)

    even on subformulas
  - Propositional logic inference rules

    whole formulas only
  - Propositional logic equivalences

    even on subformulas

Predicate Logic Proofs with more content

- In propositional logic we could just write down other propositional logic statements as "givens"

- Here, we also want to be able to use domain knowledge so proofs are about something specific

- Example:

Domain of Discourse
Integers

- Given the basic properties of arithmetic on integers, define:

Predicate Definitions
$\text{Even}(x) := \exists y\ (x = 2 \cdot y)$
$\text{Odd}(x) := \exists y\ (x = 2 \cdot y + 1)$