AND OVER THERE WE HAVE THE LABYRINTH GUARDS. ONE ALWAYS LIES, ONE ALWAYS TELLS THE TRUTH, AND ONE STABS PEOPLE WHO ASK TRICKY QUESTIONS.
Last Class: Quantifiers

We use quantifiers to talk about collections of objects.

\[ \forall x \ P(x) \]

\( P(x) \) is true for every \( x \) in the domain
read as “for all \( x, P \) of \( x \)”

\[ \exists x \ P(x) \]

There is an \( x \) in the domain for which \( P(x) \) is true
read as “there exists \( x, P \) of \( x \)”
Translate the following statements to English

∀x ∃y Greater(y, x)
  
  For every positive integer, there is a larger positive integer.

∃y ∀x Greater(y, x)
  
  There is a positive integer that is larger than every other positive integer.

∀x ∃y (Greater(y, x) ∧ Prime(y))
  
  For every positive integer, there is a prime that is larger.

Sound more natural without introducing variable names
"All red cats like tofu"

\[ \forall x ((\text{Red}(x) \wedge \text{Cat}(x)) \rightarrow \text{LikesTofu}(x)) \]

"Some red cats don’t like tofu"

\[ \exists y ((\text{Red}(y) \wedge \text{Cat}(y)) \wedge \neg \text{LikesTofu}(y)) \]
Last class: Negations of Quantifiers

Predicate Definitions

| PurpleFruit(x) ::= “x is a purple fruit” |

\((*)\) \(\forall x \\text{PurpleFruit}(x)\) (“All fruits are purple”)

What is the negation of (*)?
(a) “there exists a purple fruit”
(b) “there exists a non-purple fruit”
(c) “all fruits are not purple”

Domain of Discourse

|\{plum, apple\}|

\((*)\) \(\text{PurpleFruit}(\text{plum}) \land \text{PurpleFruit}(\text{apple})\)

(a) \(\text{PurpleFruit}(\text{plum}) \lor \text{PurpleFruit}(\text{apple})\)
(b) \(\neg \text{PurpleFruit}(\text{plum}) \lor \neg \text{PurpleFruit}(\text{apple})\)
(c) \(\neg \text{PurpleFruit}(\text{plum}) \land \neg \text{PurpleFruit}(\text{apple})\)
Last class: De Morgan’s Laws for Quantifiers

\[\neg \forall x \ P(x) \equiv \exists x \ \neg P(x)\]
\[\neg \exists x \ P(x) \equiv \forall x \ \neg P(x)\]

Intuition: \(\forall\) is like a giant AND over the domain
\(\exists\) is like a giant OR over the domain
Last class: De Morgan’s Laws for Quantifiers

\[ \neg \forall x \ P(x) \equiv \exists x \ \neg P(x) \]
\[ \neg \exists x \ P(x) \equiv \forall x \ \neg P(x) \]

These are equivalent but not equal

They have different English translations, e.g.:

There is no unicorn \( \neg \exists x \ \text{Unicorn}(x) \)

Every animal is not a unicorn \( \forall x \ \neg \text{Unicorn}(x) \)
Last class: De Morgan’s Laws for Quantifiers

\[ \neg \forall x \ P(x) \equiv \exists x \ \neg P(x) \]
\[ \neg \exists x \ P(x) \equiv \forall x \ \neg P(x) \]

“There is no integer at least as large as every other integer”

\[ \neg \exists \ x \ \forall \ y \ ( x \geq y) \]
\[ \equiv \ \forall \ x \ \neg \ \forall \ y \ ( x \geq y) \]
\[ \equiv \ \forall \ x \ \exists \ y \ \neg ( x \geq y) \]
\[ \equiv \ \forall \ x \ \exists \ y \ ( y > x) \]

“For every integer, there is a larger integer”
De Morgan’s Laws for Quantifiers

\[ \neg \forall x \, P(x) \equiv \exists x \, \neg P(x) \]
\[ \neg \exists x \, P(x) \equiv \forall x \, \neg P(x) \]

“No even prime is greater than 2”

\[ \neg \exists x \, (\text{Even}(x) \land \text{Prime}(x) \land \text{Greater}(x, 2)) \]
\[ \equiv \forall x \, \neg (\text{Even}(x) \land \text{Prime}(x) \land \text{Greater}(x, 2)) \]
\[ \equiv \forall x \, (\neg (\text{Even}(x) \land \text{Prime}(x)) \lor \neg \text{Greater}(x, 2)) \]
\[ \equiv \forall x \, ((\text{Even}(x) \land \text{Prime}(x)) \rightarrow \neg \text{Greater}(x, 2)) \]
\[ \equiv \forall x \, ((\text{Even}(x) \land \text{Prime}(x)) \rightarrow \text{LessEq}(x, 2)) \]

“Every even prime is less than or equal to 2.”
De Morgan’s Laws for Quantifiers

We just saw that

\[ \neg \exists x (P(x) \land R(x)) \equiv \forall x (P(x) \rightarrow \neg R(x)) \]

Can similarly show that

\[ \neg \forall x (P(x) \rightarrow R(x)) \equiv \exists x (P(x) \land \neg R(x)) \]

De Morgan’s Laws respect domain restrictions!
(It leaves them in place and only negates the other parts.)
De Morgan’s Laws for Quantifiers

\[ \neg \forall x \ P(x) \equiv \exists x \ \neg P(x) \]
\[ \neg \exists x \ P(x) \equiv \forall x \ \neg P(x) \]

Remain true when domain restrictions are used:

\[ \neg \exists x \ (P(x) \land R(x)) \equiv \forall x \ (P(x) \rightarrow \neg R(x)) \]
\[ \neg \forall x \ (P(x) \rightarrow R(x)) \equiv \exists x \ (P(x) \land \neg R(x)) \]
Scope of Quantifiers

\[ \exists x \ (P(x) \land Q(x)) \quad \text{vs.} \quad (\exists x \ P(x)) \land (\exists x \ Q(x)) \]
### Scope of Quantifiers

<table>
<thead>
<tr>
<th>Quantifier Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exists x \ (P(x) \land Q(x)) ) vs. ( (\exists x \ P(x)) \land (\exists x \ Q(x)) )</td>
<td>This one asserts ( P ) and ( Q ) of the same ( x ). This one asserts ( P ) and ( Q ) of potentially different ( x )'s.</td>
</tr>
</tbody>
</table>
### Scope of Quantifiers

**Example:**  

\[ \text{NotLargest}(x) \equiv \exists y \, \text{Greater}(y, x) \]  
\[ \equiv \exists z \, \text{Greater}(z, x) \]

**truth value:**  
- doesn’t depend on \( y \) or \( z \) “bound variables”  
- does depend on \( x \) “free variable”

Quantifiers only act on **free** variables of the formula they quantify.

\[ \forall x \, (\exists y \, (P(x,y) \rightarrow \forall x \, Q(y, x))) \]
Quantifier “Style”

\[ \forall x (\exists y \ (P(x,y) \rightarrow \forall x \ Q(y,x))) \]

This isn’t “wrong”, it’s just horrible style. Don’t confuse your reader by using the same variable multiple times...there are a lot of letters...
Nested Quantifiers

• Quantified variable names don’t matter

\[ \forall x \exists y \, P(x, y) \equiv \forall a \exists b \, P(a, b) \]

• Positions of quantifiers can sometimes change

\[ \forall x \, (Q(x) \land \exists y \, P(x, y)) \equiv \forall x \, \exists y \, (Q(x) \land P(x, y)) \]

• But: order is important...
Quantifier Order Can Matter

<table>
<thead>
<tr>
<th>Domain of Discourse</th>
<th>Predicate Definitions</th>
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</thead>
<tbody>
<tr>
<td>{1, 2, 3, 4}</td>
<td>GreaterEq(x, y) ::= “x ≥ y”</td>
</tr>
</tbody>
</table>

“There is a number greater than or equal to all numbers.”

\[ \exists x \ \forall y \ \text{GreaterEq}(x, y) \]

```
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
T & F & F & F \\
T & T & F & F \\
T & T & T & F \\
T & T & T & T \\
\end{array}
```
Quantifier Order Can Matter

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“There is a number greater than or equal to all numbers.”

\[ \exists x \ \forall y \ \text{GreaterEq}(x, y)) \]

“Every number has a number greater than or equal to it.”

\[ \forall y \ \exists x \ \text{GreaterEq}(x, y)) \]
Quantifier Order Can Matter

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“There is a number greater than or equal to all numbers.”

\[\exists x \ \forall y \ GreaterEq(x, y)\]  

“Every number has a number greater than or equal to it.”

\[\forall y \ \exists x \ GreaterEq(x, y)\]  

The purple statement requires **an entire row** to be true.  
The red statement requires one entry in **each column** to be true.

**Important:** both include the case \(x = y\)  
**Different names does not imply different objects!**
## Quantification with Two Variables

<table>
<thead>
<tr>
<th>expression</th>
<th>when true</th>
<th>when false</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \forall y \ P(x, y)$</td>
<td>Every pair is true.</td>
<td>At least one pair is false.</td>
</tr>
<tr>
<td>$\exists x \exists y \ P(x, y)$</td>
<td>At least one pair is true.</td>
<td>All pairs are false.</td>
</tr>
<tr>
<td>$\forall x \exists y \ P(x, y)$</td>
<td>We can find a specific $y$ for each $x$. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$</td>
<td>Some $x$ doesn’t have a corresponding $y$.</td>
</tr>
<tr>
<td>$\exists y \forall x \ P(x, y)$</td>
<td>We can find ONE $y$ that works no matter what $x$ is. $(x_1, y), (x_2, y), (x_3, y)$</td>
<td>For any candidate $y$, there is an $x$ that it doesn’t work for.</td>
</tr>
</tbody>
</table>
Logical Inference

• So far we’ve considered:
  – How to understand and *express* things using propositional and predicate logic
  – How to *compute* using Boolean (propositional) logic
  – How to show that different ways of expressing or computing them are *equivalent* to each other

• Logic also has methods that let us *infer* implied properties from ones that we know
  – Equivalence is a small part of this
New Perspective

Rather than comparing $A$ and $B$ as columns, zoom in on just the rows where $A$ is true:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td></td>
<td>$F$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td></td>
<td>$F$</td>
</tr>
</tbody>
</table>
New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where A is true:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>

Given that A is true, we see that B is also true.

A ⇒ B
New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where A is true:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>?</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>?</td>
</tr>
</tbody>
</table>

When we zoom out, what have we proven?
New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where B is true:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>A</th>
<th>B</th>
<th>A → B</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

When we zoom out, what have we proven?

\[ A \Rightarrow B \]

\[ (A \rightarrow B) \equiv T \]

\[ A \equiv B \]

\[ (A \leftrightarrow B) \equiv T \]
New Perspective

Equivalences

\[ A \equiv B \text{ and } (A \leftrightarrow B) \equiv T \text{ are the same} \]

Inference

\[ A \Rightarrow B \text{ and } (A \rightarrow B) \equiv T \text{ are the same} \]

Can do the inference by zooming in to the rows where \( A \) is true
Applications of Logical Inference

- **Software Engineering**
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied

- **Artificial Intelligence**
  - Automated reasoning

- **Algorithm design and analysis**
  - e.g., Correctness, Loop invariants.

- **Logic Programming, e.g. Prolog**
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution
Proofs

• Start with given facts (hypotheses)
• Use rules of inference to extend set of facts
• Result is proved when it is included in the set
An inference rule: *Modus Ponens*

- If $A$ and $A \rightarrow B$ are both true, then $B$ must be true.

Write this rule as:

$\frac{A \land A \rightarrow B}{\therefore B}$

Given:
- If it is Friday, then you have a 311 class today.
- It is Friday.

Therefore, by Modus Ponens:
- You have a 311 class today.
My First Proof!

Show that \( r \) follows from \( p, p \rightarrow q, \) and \( q \rightarrow r \)

1. \( p \) Given
2. \( p \rightarrow q \) Given
3. \( q \rightarrow r \) Given
4. \( q \) Modus Ponens: 1, 2
5. \( r \) Modus Ponens: 3, 4

Modus Ponens: \( A ; A \rightarrow B \quad \therefore B \)
My First Proof!

Show that \( r \) follows from \( p, p \rightarrow q, \) and \( q \rightarrow r \)

1. \( p \) \hspace{1cm} \text{Given}
2. \( p \rightarrow q \) \hspace{1cm} \text{Given}
3. \( q \rightarrow r \) \hspace{1cm} \text{Given}
4. \( q \) \hspace{1cm} \text{MP: 1, 2}
5. \( r \) \hspace{1cm} \text{MP: 3, 4}

Modus Ponens

\[
\begin{array}{c}
A ; A \rightarrow B \\
\therefore B
\end{array}
\]
Proofs can use equivalences too

Show that \( \neg p \) follows from \( p \to q \) and \( \neg q \)

1. \( p \to q \) \hspace{1cm} \text{Given}
2. \( \neg q \) \hspace{1cm} \text{Given}
3. \( \neg q \to \neg p \) \hspace{1cm} \text{Contrapositive: 1}
4. \( \neg p \) \hspace{1cm} \text{MP: 2, 3}

Modus Ponens \( A ; A \to B \)

\[ \therefore B \]
Inference Rules

Requirements: \( A \land B \)

Conclusions: \( \therefore C, D \)

Then, \( C \) must be true

Then \( D \) must be true

Example (Modus Ponens):

\[
\begin{align*}
A \land A \to B \\
\therefore B
\end{align*}
\]

If I have \( A \) and \( A \to B \) both true, then \( B \) must be true.
Axioms: Special inference rules

If I have nothing...

Requirements:

Conclusions: \( \therefore C, D \)

Then, \( C \) must be true
Then, \( D \) must be true

Example (Excluded Middle):

\[ \therefore A \lor \neg A \]

\( A \lor \neg A \) must be true.
Simple Propositional Inference Rules

Two inference rules per binary connective, one to **eliminate** it and one to **introduce** it

\[ A \land B \quad \therefore \quad A, B \]

\[ A ; B \quad \therefore \quad A \land B \]

\[ A \lor B ; \neg A \quad \therefore \quad B \]

\[ A \quad \therefore \quad A \lor B, B \lor A \]

\[ A \Rightarrow B \quad \therefore \quad A \Rightarrow B \]

Modus Ponens

Elim \( \land \)

Intro \( \land \)

Elim \( \lor \)

Intro \( \lor \)

Direct Proof

Not like other rules
Proofs

Show that \( r \) follows from \( p, p \rightarrow q \) and \((p \land q) \rightarrow r\)

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

1. \( p \) \hspace{0.5cm} \text{given}
2. \( p \rightarrow q \) \hspace{0.5cm} \text{given}
3. \((p \land q) \rightarrow r\) \hspace{0.5cm} \text{given}
4. \( q \) \hspace{0.5cm} \text{mp:1,2}
5. \( p \land q \) \hspace{0.5cm} \text{Intro } \land : 1,4

\[ A ; A \rightarrow B \]
\[ \therefore B \]

\[ A \land B \]
\[ \therefore A, B \]

\[ \text{Intro } \land \]

\[ A ; B \]
\[ \therefore A \land B \]
Proofs

Show that \( r \) follows from \( p \), \( p \rightarrow q \), and \( p \land q \rightarrow r \)

1. \( p \)  
   - Given
2. \( p \rightarrow q \)  
   - Given
3. \( q \)  
   - MP: 1, 2
4. \( p \land q \)  
   - Intro \( \land \): 1, 3
5. \( p \land q \rightarrow r \)  
   - Given
6. \( r \)  
   - MP: 4, 5

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that’s great!