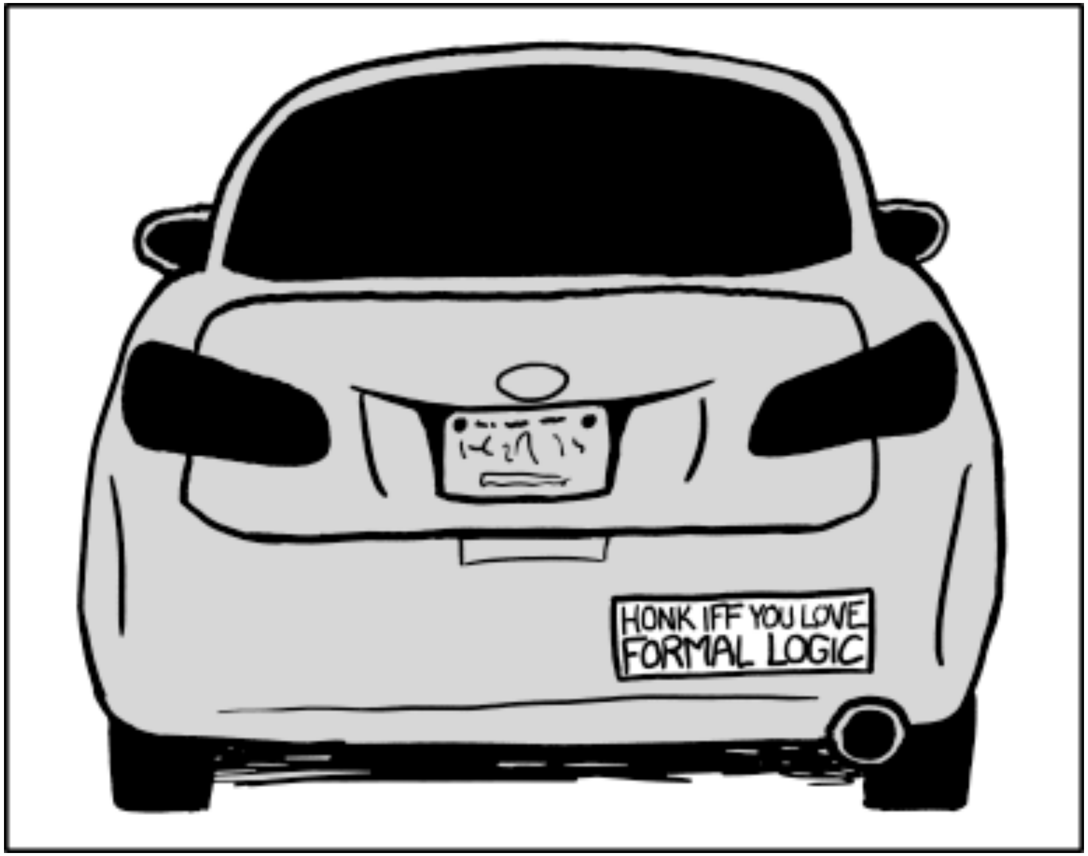
CSE 311: Foundations of Computing I

Lecture 1: Propositional Logic



What is logic and why do we need it?

Logic is a language, like English or Java, with its own

- words and rules for combining words into sentences (syntax)
- ways to assign meaning to words and sentences (semantics)

Why learn another language?

We know English and Java already?

Why not use English?

- Turn right here…
- We saw her duck
- Buffalo buffalo Buffalo buffalo buffalo buffalo Buffalo buffalo

Natural languages can be unclear or imprecise

Why not use English?

– Turn right here…

Does “right” mean the direction or now?

– We saw her duck

Does “duck” mean the animal or crouch down?

– Buffalo buffalo Buffalo buffalo buffalo buffalo Buffalo buffalo

This means “Bison from Buffalo, that bison from Buffalo bully, themselves bully bison from Buffalo.

Natural languages can be unclear or imprecise

Why learn a new language?

We need a language of reasoning to

- state sentences more precisely
- state sentences more concisely
- understand sentences more quickly

Formal logic has these properties

Propositions: building blocks of logic

A ***proposition*** is a statement that

- is either true or false
- is “well-formed”

Propositions: building blocks of logic

A ***proposition*** is a statement that

- is either true or false
- is "well-formed"

All cats are mammals

true

All mammals are cats

false

Are These Propositions?

2 + 2 = 5

x + 2 = 5389, where x is my PIN number

Akjsdf!

Who are you?

Every positive even integer can be written as the sum of two primes.

Are These Propositions?

2 + 2 = 5

This is a proposition. It's okay for propositions to be false.

x + 2 = 5389, where x is my PIN number

This is a proposition. We don't need to know what x is.

Akjsdf!

Not a proposition because it's gibberish.

Who are you?

This is a question which means it doesn't have a truth value.

Every positive even integer can be written as the sum of two primes.

This is a proposition. We don't know if it's true or false, but we know it's one of them!

Propositions

We need a way of talking about *arbitrary* ideas…

Propositional Variables: p, q, r, s, …

Truth Values:

- **T** for **true**
- **F** for **false**

Familiar from Java

- Java `boolean` represents a truth value
  - constants `true` and `false`
  - variables hold *unknown* values
- Operators that calculate new truth values from given ones
  - unary: not (`!`)
  - binary: and (&&), or (`||`)

A Proposition

"You can get measles and mumps if you didn't have the MMR vaccine, but if you had the MMR vaccine then you can't get either measles or mumps."

We'd like to *understand* what this proposition means.

This is where logic comes in. There are pieces that appear multiple times in the phrase (e.g., "you can get measles").

These are called ***atomic propositions***. Let's list them:

p: "You can get measles"

q: "You can get mumps"

r: "You had the MMR vaccine"

Vaccine Sentence is a *Compound* Proposition

"You can get measles and mumps if you didn't have the MMR vaccine, but if you had the MMR vaccine then you can't get either measles or mumps."

p: "You can get measles"

q: "You can get mumps"

r: "You had the MMR vaccine"

Now, we see how they fit together to make the sentence:

((p and q) if not r) but (if r then not (p or q))

"but" is just an unexpected "and"

((p and q) if not r) and (if r then not (p or q))

To fully translate to formal language we need *connectives*

Logical Connectives

Negation (not)	$\neg p$
Conjunction (and)	$p \land q$
Disjunction (or)	$p \lor q$
Exclusive Or	$p \oplus q$
Implication	$p \longrightarrow q$
Biconditional	$p \leftrightarrow q$

These build new propositions from simpler ones

- The truth values for these new propositions are given by *truth tables*.

Some Truth Tables

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logic forces us to distinguish $\vee$ from $\oplus$

Implication

"If it's raining, then I have my umbrella"

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

It's useful to think of implications as promises. That is "Did I lie?"

	It's raining	It's not raining
I have my umbrella		
I do not have my umbrella		

Implication

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

"If it's raining, then I have my umbrella"

It's useful to think of implications as promises. That is "Did I lie?"

	It's raining	It's not raining
I have my umbrella	No	No
I do not have my umbrella	**Yes**	No

The only lie is when:

(a) It's raining AND

(b) I don't have my umbrella

Implication

"If the Seahawks won, then I was at the game."

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

What's the one scenario where I lied?

	I was at the game	I wasn't at the game
Seahawks won		
Seahawks lost		

Implication

"If the Seahawks won, then I was at the game."

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

What's the one scenario where I lied?

	I was at the game	I wasn't at the game
Seahawks won	Ok	**I lied**
Seahawks lost	Ok	Ok

Implication

"If it's raining, then I have my umbrella"

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Are these true?

$2 + 2 = 4 \rightarrow$ *earth is a planet*

$2 + 2 = 5 \rightarrow$ *26 is prime*

Implication is not a causal relationship!

Implication

"If it's raining, then I have my umbrella"

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Are these true?

$2 + 2 = 4 \to$ *earth is a planet*

The fact that these are unrelated doesn't make the statement false! "2 + 2 = 4" is true; "earth is a planet" is true. T→T is true. So, the statement is true.

$2 + 2 = 5 \to$ *26 is prime*

Again, these statements may or may not be related. "2 + 2 = 5" is false; so, the implication is true. (Whether 26 is prime or not is irrelevant).

Implication is not a causal relationship!

$p \rightarrow q$

(1) "I have a billion dollars if I am a billionaire"

(2) "I have a billion dollars only if I am a billionaire"

These sentences are implications in opposite directions:

(1) "Billionaires must have a billion dollars"

(2) "People who have a billion dollars are billionaires"

So, the implications are:

(1) If I am a billionaire, *then* I have a billion dollars.

(2) If I have a billion dollars, *then* I am a billionaire.

$p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication:

- p implies q
- whenever p is true q must be true
- if p then q
- q if p
- p is sufficient for q
- p only if q
- *q is necessary for p*

Biconditional: $p \leftrightarrow q$

- *p* if and only if *q* (*p* iff *q*)
- *p* is true exactly when *q is true*
- *p* implies *q* and *q* implies *p*
- *p* is necessary and sufficient for *q*

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Back to the Vaccine Sentence...

Negation (not)	$\neg p$
Conjunction (and)	$p \wedge q$
Disjunction (or)	$p \vee q$
Exclusive Or	$p \oplus q$
Implication	$p \longrightarrow q$
Biconditional	$p \leftrightarrow q$

p : "You can get measles"

q : "You can get mumps"

r : "You had the MMR vaccine"

"You can get measles and mumps if you didn't have the MMR vaccine, but if you had the MMR vaccine then you can't get either measles or mumps."

((p and q) if not r) $\wedge$ (if r then not (p or q))

(($p \wedge q$) if $\neg r$) $\wedge$ (if r then $\neg(p \vee q)$)

$$(\neg r \rightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$$

Analyzing the Vaccine Sentence with a Truth Table

$$(\neg r \to (p \land q)) \land (r \to \neg(p \lor q))$$

p	q	r	$\left(\neg r \longrightarrow (p \land q)\right) \land (r \to \neg(p \lor q))$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Analyzing the Vaccine Sentence with a Truth Table

$$(\neg r \rightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$$

p	q	r	$\neg r \longrightarrow (p \wedge q)$	$r \rightarrow \neg(p \vee q)$	$(\neg r \longrightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Analyzing the Vaccine Sentence with a Truth Table

$$(\neg r \to (p \land q)) \land (r \to \neg(p \lor q))$$

p	q	r	$\neg r$	$p \land q$	$\neg r \longrightarrow (p \land q)$	$r \to \neg(p \lor q)$	$(\neg r \longrightarrow (p \land q)) \land (r \to \neg(p \lor q))$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Analyzing the Vaccine Sentence with a Truth Table

$$(\neg r \rightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$$

p	q	r	$\neg r$	$p \wedge q$	$\neg r \longrightarrow (p \wedge q)$	$\neg(p \vee q)$	$r \rightarrow \neg(p \vee q)$	$(\neg r \longrightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Analyzing the Vaccine Sentence with a Truth Table

$$(\neg r \rightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$$

p	q	r	$\neg r$	$p \wedge q$	$\neg r \longrightarrow (p \wedge q)$	$p \vee q$	$\neg(p \vee q)$	$r \rightarrow \neg(p \vee q)$	$(\neg r \longrightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$
T	T	T							
T	T	F							
T	F	T							
T	F	F							
F	T	T							
F	T	F							
F	F	T							
F	F	F							

Analyzing the Vaccine Sentence with a Truth Table

$(\neg r \rightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$

p	q	r	$\neg r$	$p \wedge q$	$\neg r \longrightarrow (p \wedge q)$	$p \vee q$	$\neg(p \vee q)$	$r \rightarrow \neg(p \vee q)$	$(\neg r \longrightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$
T	T	T	F	T		T			
T	T	F	T	T		T			
T	F	T	F	F		T			
T	F	F	T	F		T			
F	T	T	F	F		T			
F	T	F	T	F		T			
F	F	T	F	F		F			
F	F	F	T	F		F			

Analyzing the Vaccine Sentence with a Truth Table

$$(\neg r \rightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$$

p	q	r	$\neg r$	$p \wedge q$	$\neg r \longrightarrow (p \wedge q)$	$p \vee q$	$\neg(p \vee q)$	$r \rightarrow \neg(p \vee q)$	$(\neg r \longrightarrow (p \wedge q)) \wedge (r \rightarrow \neg(p \vee q))$
T	T	T	F	T	T	T	F	F	F
T	T	F	T	T	T	T	F	T	T
T	F	T	F	F	T	T	F	F	F
T	F	F	T	F	F	T	F	T	F
F	T	T	F	F	T	T	F	F	F
F	T	F	T	F	F	T	F	T	F
F	F	T	F	F	T	F	T	T	T
F	F	F	T	F	F	F	T	T	F

Biconditional: $p \leftrightarrow q$

- *p* if and only if *q* (*p* iff *q*)
- *p* is true exactly when *q is true*
- *p* implies *q* and *q* implies *p*
- *p* is necessary and sufficient for *q*

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	
T	F	F	F	T	
F	T	F	T	F	
F	F	T	T	T	

Biconditional: $p \leftrightarrow q$

- ***p*** **if and only if *q*** *(**p** **iff** **q**)*
- ***p*** **is true exactly when *q is true***
- ***p*** **implies *q* and *q* implies *p***
- ***p*** **is necessary and sufficient for *q***

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T