CSE 311 Section 6

Induction
Administrivia
Announcements & Reminders

● HW4
  ○ Grades out now
  ○ If you think something was graded incorrectly, submit a regrade request!

● HW5 (BOTH PARTS)
  ○ BOTH PARTS due Wednesday 10/8 @ 10pm

● Midterm is Coming!!!
  ○ Wednesday 10/15 @ 6-7:30 pm in BAG 131 and 154
  ○ If you cannot make it, please let us know ASAP and we will schedule you for a makeup
Induction
(Weak) Induction Template

Let $P(n)$ be “(whatever you’re trying to prove)”. We show $P(n)$ holds for all $n$ by induction on $n$.

**Base Case:** Show $P(b)$ is true.

**Inductive Hypothesis:** Suppose $P(k)$ holds for an arbitrary $k \geq b$.

**Inductive Step:** Show $P(k + 1)$ (i.e. get $P(k) \rightarrow P(k + 1)$)

**Conclusion:** Therefore, $P(n)$ holds for all $n$ by the principle of induction.
(Weak) Induction Template

Let $P(n)$ be “(whatever you’re trying to prove)”. We show $P(n)$ holds for all $n$ by induction on $n$.

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k + 1)$ (i.e. get $P(k) \rightarrow P(k + 1)$)

Conclusion: Therefore, $P(n)$ holds for all $n$ by the principle of induction.

Note: often you will condition $n$ here, like “all natural numbers $n$” or “$n \geq 0$”.

Match the earlier condition on $n$ in your conclusion!
Problem 1 – Induction with Equality

a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

Let's walk through part (a) together.

We can “fill in” our induction template to construct our proof by induction.
Problem 1 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be “”. We show $P(n)$ holds for (some) $n$ by induction on $n$.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k + 1)$:

Conclusion: Therefore, $P(n)$ holds for (some) $n$ by the principle of induction.
Problem 1 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be “$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.” We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on $n$.

Base Case: $P(b)$:
Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.
Inductive Step: Goal: Show $P(k + 1)$:

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.
Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be “$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$”. We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on $n$.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k + 1)$:

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.
Let $P(n)$ be “$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$”. We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on $n$.

**Base Case:** $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

**Inductive Hypothesis:** Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

**Inductive Step:** Goal: Show $P(k+1)$:

**Conclusion:** Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.
Problem 1 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be “$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$”. We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on $n$.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Inductive Step: Goal: Show $P(k + 1)$: $0 + 1 + \cdots + k + (k + 1) = \frac{(k+1)(k+2)}{2}$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.
Problem 1 – Induction with Equality

Show using induction that

\[0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\]

for all \(n \in \mathbb{N}\).

Let \(P(n)\) be “\(0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\).” We show \(P(n)\) holds for all \(n \in \mathbb{N}\) by induction on \(n\).

Base Case: \(P(0)\): \(0 + \cdots + 0 = \frac{0(0+1)}{2}\) so the base case holds.

Inductive Hypothesis: Suppose \(P(k)\) holds for an arbitrary \(k \geq 0\), i.e. \(0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}\).

Inductive Step: Goal: Show \(P(k + 1)\): \(0 + 1 + \cdots + k + (k + 1) = \frac{(k+1)(k+2)}{2}\)

\[
0 + 1 + \cdots + k + (k + 1) = \cdots
\]

\[
= \frac{(k+1)(k+2)}{2}
\]

Conclusion: Therefore, \(P(n)\) holds for all \(n \in \mathbb{N}\) by the principle of induction.
Let $P(n)$ be “$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$”. We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on $n$.

**Base Case:** $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

**Inductive Hypothesis:** Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

**Inductive Step:** Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k + 1) = \frac{(k+1)(k+2)}{2}$

$0 + 1 + \cdots + k + (k + 1) = (0 + 1 + \cdots + k) + (k + 1)$

$\cdots$

$= \frac{(k+1)(k+2)}{2}$

**Conclusion:** Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.
Problem 1 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on $n$.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k + 1) = \frac{(k+1)(k+2)}{2}$

$0 + 1 + \cdots + k + (k + 1) = (0 + 1 + \cdots + k) + (k + 1)$

$= \frac{k(k+1)}{2} + (k + 1)$ by I.H.

$\cdots$

$= \frac{(k+1)(k+2)}{2}$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.
Problem 1 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be “$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.” We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on $n$.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k + 1) = \frac{(k+1)(k+2)}{2}$

$0 + 1 + \cdots + k + (k + 1) = (0 + 1 + \cdots + k) + (k + 1)$

$= \frac{k(k+1)}{2} + (k + 1)$ by l.H.

$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.
Show using induction that
\[ 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \]
for all \( n \in \mathbb{N} \).

**Problem 1 – Induction with Equality**

Let \( P(n) \) be “\( 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \)”. We show \( P(n) \) holds for all \( n \in \mathbb{N} \) by induction on \( n \).

**Base Case:** \( P(0) \): \( 0 + \cdots = 0 = \frac{0(0+1)}{2} \) so the base case holds.

**Inductive Hypothesis:** Suppose \( P(k) \) holds for an arbitrary \( k \geq 0 \), i.e. \( 0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2} \).

**Inductive Step:** Goal: Show \( P(k+1) \): \( 0 + 1 + \cdots + k + (k + 1) = \frac{(k+1)(k+2)}{2} \)

\[
0 + 1 + \cdots + k + (k + 1) = (0 + 1 + \cdots + k) + (k + 1)
\]
\[
= \frac{k(k+1)}{2} + (k + 1) \\
\text{by I.H.}
\]
\[
= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}
\]
\[
= \frac{k(k+1) + 2(k+1)}{2}
\]
\[
= \frac{(k+1)(k+2)}{2}
\]

**Conclusion:** Therefore, \( P(n) \) holds for all \( n \in \mathbb{N} \) by the principle of induction.
Problem 1 – Induction with Equality

Show using induction that 
\[0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\]
for all \(n \in \mathbb{N}\).

Let \(P(n)\) be “\(0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\)”. We show \(P(n)\) holds for all \(n \in \mathbb{N}\) by induction on \(n\).

**Base Case:** \(P(0)\): \(0 + \cdots = 0 = \frac{0(0+1)}{2}\) so the base case holds.

**Inductive Hypothesis:** Suppose \(P(k)\) holds for an arbitrary \(k \geq 0\), i.e. \(0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}\)

**Inductive Step:** Goal: Show \(P(k + 1)\): \(0 + 1 + \cdots + k + (k + 1) = \frac{(k+1)(k+2)}{2}\)
\[
0 + 1 + \cdots + k + (k + 1) = (0 + 1 + \cdots + k) + (k + 1)
= \frac{k(k+1)}{2} + (k + 1)
= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}
= \frac{k(k+1)+2(k+1)}{2}
= \frac{(k+1)(k+2)}{2}
\]

by l.H.

factoring out \((k+1)\)

**Conclusion:** Therefore, \(P(n)\) holds for all \(n \in \mathbb{N}\) by the principle of induction.
Problem 1 – Induction with Equality

a) Show using induction that \(0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\) for all \(n \in \mathbb{N}\).

b) Define the triangle numbers as \(\triangle_n = 1 + 2 + \cdots + n\), where \(n \in \mathbb{N}\). In part (a) we showed \(\triangle_n = \frac{n(n+1)}{2}\). Prove the following equality for all \(n \in \mathbb{N}\):

\[0^3 + 1^3 + \cdots + n^3 = \triangle_n^2\]

Now try part (b) with people around you, and then we’ll go over it together!
Problem 1 – Induction with Equality

Let $P(n)$ be “”. We show $P(n)$ holds for (some) $n$ by induction on $n$.

**Base Case:** $P(b)$:

**Inductive Hypothesis:** Suppose $P(k)$ holds for an arbitrary $k \geq b$.

**Inductive Step:** Goal: Show $P(k + 1)$:

**Conclusion:** Therefore, $P(n)$ holds for (some) $n$ by the principle of induction.

$\Delta_n = 1 + 2 + \cdots + n, \; n \in \mathbb{N}.$

$\Delta_n = \frac{n(n+1)}{2}$. Prove for all $n \in \mathbb{N}$:

$0^3 + 1^3 + \cdots + n^3 = \Delta_n^2$
Strong Induction
Why Strong Induction?

In **weak induction**, the inductive hypothesis only assumes that $P(k)$ is true and uses that in the inductive step to prove the implication $P(k) \rightarrow P(k + 1)$.

In **strong induction**, the inductive hypothesis assumes the predicate holds for every step from the base case(s) up to $P(k)$. This usually looks something like $P(b_1) \wedge P(b_2) \wedge \cdots \wedge P(k)$. Then it uses this stronger inductive hypothesis in the inductive step to prove the implication $P(b_1) \wedge \cdots \wedge P(k) \rightarrow P(k + 1)$.

Strong induction is necessary when we have multiple base cases, or when we need to go back to a smaller number than $k$ in our inductive step.
Strong Induction Template

Let $P(n)$ be “(whatever you’re trying to prove)”. We show $P(n)$ holds for all $n \geq b_{min}$ by induction on $n$.

**Base Case:** Show $P(b_{min}), P(b_{min}+1), \ldots, P(b_{max})$ are all true.

**Inductive Hypothesis:** Suppose $P(b_{min}) \land \cdots \land P(k)$ hold for an arbitrary $k \geq b_{max}$.

**Inductive Step:** Show $P(k + 1)$ (i.e. get $P(b_{min}) \land \cdots \land P(k) \rightarrow P(k + 1)$)

**Conclusion:** Therefore, $P(n)$ holds for all $n \geq b_{min}$ by the principle of induction.
Problem 4 – Cantelli’s Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function $f$:

\[
\begin{align*}
  f(0) &= 0 \\
  f(1) &= 1 \\
  f(n) &= 2f(n - 1) - f(n - 2) \text{ for } n \geq 2
\end{align*}
\]

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year $n$. That is, construct a formula for $f(n)$ and prove its correctness.

First, let’s construct a formula for $f(n)$. How many rabbits does he have each year? Let’s do some calculations, and see if we can find a pattern. Then, we’ll use induction to prove the pattern holds for all $n$!
Problem 4 – Cantelli’s Rabbits

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\[
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Problem 4 – Cantelli’s Rabbits

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$$f(0) = 0$$
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Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year $n$. That is, construct a formula for $f(n)$ and prove its correctness.

$$f(0) = 0$$
$$f(1) = 1$$
$$f(2) = 2f(2 - 1) - f(2 - 2) = 2f(1) - f(0) = 2(1) - 0 = 2 - 0 = 2$$
$$f(3) = 2f(3 - 1) - f(3 - 2) = 2f(2) - f(1) = 2(2) - 1 = 4 - 1 = 3$$
$$f(4) = 2f(4 - 1) - f(4 - 2) = 2f(3) - f(2) = 2(3) - 2 = 6 - 2 = 4$$

It seems like we have a pattern here!

$$f(n) = n$$

But we don’t want to have to check for EVERY $n$, so let’s see if we can prove it with induction instead!
Problem 4 – Cantelli’s Rabbits

What kind of induction should we use?
Problem 4 – Cantelli’s Rabbits

What kind of induction should we use?

Strong induction!
Problem 4 – Cantelli’s Rabbits

What kind of induction should we use?

Strong induction!

Two big clues:

- Multiple base cases in the formula: \( f(0) = 0 \) and \( f(1) = 1 \)
- Recursively defined step of formula goes back further than just \( n \):
  - \( f(n) \) based on both \( f(n - 1) \) and \( f(n - 2) \)
  - for \( P(n) \) to be true, both \( P(n - 1) \) and \( P(n - 2) \) must be true
Let $P(n)$ be “(whatever you’re trying to prove)”. 
We show $P(n)$ holds for all $n \geq b_{min}$ by induction on $n$.

**Base Case:** Show $P(b_{min})$, $P(b_{min}+1)$, ..., $P(b_{max})$ are all true.

**Inductive Hypothesis:** Suppose $P(b_{min}) \land \cdots \land P(k)$ hold for an arbitrary $k \geq b_{max}$.

**Inductive Step:** Show $P(k + 1)$ (i.e. get $P(b_{min}) \land \cdots \land P(k) \rightarrow P(k + 1)$)

**Conclusion:** Therefore, $P(n)$ holds for all $n \geq b_{min}$ by the principle of induction.

Fill in the strong induction template to prove the claim!
Problem 4 – Cantelli’s Rabbits

Let $P(n)$ be “”.
We show $P(n)$ holds …

Base Cases:

Inductive Hypothesis:

Inductive Step:

Conclusion: Therefore, $P(n)$ holds for all … by the principle of induction.
That’s All, Folks!

Thanks for coming to section this week!
Any questions?