Let $P(A)$ be “There is an NFA whose language is the same as the language for $A$.”

Let $R$ be a regex not covered by the base cases. By the exclusion rule, $R = A \cup B$ or $AB$ or $A^*$ from some regexes $A, B$

Inductive Hypothesis: Suppose $P(A)$ and $P(B)$.

Inductive Step: Case 2: $AB$

Want a machine that accepts exactly strings matched by $AB$.

Forcing a Mistake

How do we know $x, y$ must be in different states?
Well if one would be accepted and the other rejected, that would be a clear sign.

Or if there’s some string $z$ where $xz$ is accepted but $yz$ is rejected (or vice versa).

The machine is deterministic! If $x$ and $y$ take you to the same state, then $xz$ and $yz$ are also in the same state!
A Proof Outline

Claim: \( \{0^k1^k: k \geq 0\} \) is an irregular language.

...  

Let \( S = [\text{TODO}] \). \textit{S is an infinite set of strings.}  

Because the DFA is finite, there are two (different) strings \( x, y \) in \( S \) such that \( x \) and \( y \) go to the same state. \textit{We don't get to choose} \( x, y \).  

Consider the string \( z = [\text{TODO}] \). \textit{We do get to choose} \( z \) depending on \( x, y \).  

Since \( x, y \) led to the same state and \( M \) is deterministic, \( xz \) and \( yz \) will also lead to the same state \( q \) in \( M \). Observe that \( xz \in \{0^k1^k: k \geq 0\} \) but \( yz \notin \{0^k1^k: k \geq 0\} \). Since \( q \) is can be only one of an accept or reject state, \( M \) does not actually recognize \( \{0^k1^k: k \geq 0\} \). That's a contradiction!  

Therefore, \( \{0^k1^k: k \geq 0\} \) is an irregular language.