Trying a direct proof

∀a(Even(a²)→Even(a)) “if a² is even, then a is even.”

Proof By Contradiction

Claim: √2 is irrational (i.e. not rational).

Proof:
Suppose for the sake of contradiction that √2 is rational.
By definition of rational, there are integers s, t such that t ≠ 0 and √2 = s/t. Without loss of generality, let s/t be in lowest terms (i.e., with no common factors greater than 1).

\[ \sqrt{2} = \frac{s}{t} \]

\[ 2 = \frac{s^2}{t^2} \]

2t² = s² so s² is even. By the fact above, s is even, i.e. s = 2k for some integer k. Squaring both sides s² = 4k²

Substituting into our original equation, we have: 2t² = 4k², i.e. t² = 2k².
So t² is even (by definition of even). Applying the fact above again, t is even.
But if both s and t are even, they have a common factor of 2. But we said the fraction was in lowest terms.
That’s a contradiction! We conclude √2 is irrational.
What’s the difference?

What’s the difference between proof by contrapositive and proof by contradiction?

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Another Proof By Contradiction

Claim: There are infinitely many primes.

Proof:

Suppose for the sake of contradiction, that there are only finitely many primes. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdot \ldots \cdot p_k + 1$.

Case 1: $q$ is prime

Case 2: $q$ is composite

But [] is a contradiction! So there must be infinitely many primes.