Warm up translate to predicate logic:
"For every x, if x is prime, then x is odd or $x = 2$."

Nested Unalike Quantifiers

CSE 311 Fall 23
Lecture 6

Announcements

HW2 Problem 2 has a bug.

We’re working on how to fix it (probably clarification/an extra hint on part c).

We’ll send everyone an email on Ed (and update the pdf on the webpage) when it’s fixed.

Today

More on quantifiers

What happens when we want to talk about just part of our domain of discourse?

$\forall, \exists$ in the same sentence

How do we negate a quantified sentence?

Where were we?

A predicate is a function that outputs a Boolean

`Prime(x)` := “`x` is prime”

`LessThan(x,y)`:= “`x<y`”

The “domain of discourse” is the set of all values your variables can take.

Usually the “type” you’re allowing

Quantifiers

We have two extra symbols to indicate which way we're using the variable.

1. The statement is true for every x, we just want to put a name on it.

$\forall x\ (\mathrm{p}(\mathrm{x}) \wedge q(x))$ means "for every x in our domain, $p(x)$ and $q(x)$ both evaluate to true."

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, such that $p(x)$ and $q(x)$ are both true.

Quantifiers

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$\forall x\ (\mathrm{p}(\mathrm{x}) \wedge q(x))$ means "for every x in our domain, $p(x)$ and $q(x)$ both evaluate to true."

Universal Quantifier

"$\forall x$"

"for each x", "for every x", "for all x" are common translations

Remember: upside-down-A for All.

Quantifiers

Existential Quantifier

“$\exists x$“

“there is an x”, “there exists an x”, “for some x” are common translations

Remember: backwards-E for Exists.

2. There’s some x out there that works, (but I might not know which it is, so I’m using a variable).

$\exists x(p(x) \wedge q(x))$ means “there is an x in our domain, for which $p(x)$ and $q(x)$ are both true.

Translations

"For every x, if x is even, then $x = 2$."

"There are x, y such that $\mathrm{x} < y$."

$\exists x\ (\mathtt{Odd}(x) \wedge \mathtt{LessThan}(x, 5))$

$\forall y\ (\mathrm{Even}(y) \wedge \mathrm{Odd}(y))$

pollev.com/robbie

Help me adjust my explanation!

Translations

"For every x, if x is even, then $x = 2$."

$\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

"There are x, y such that $\mathrm{x} < y$."

$\exists x \exists y(\texttt{LessThan}(x, y))$

$\exists x\ (\texttt{Odd}(x) \wedge \texttt{LessThan}(x, 5))$

There is an odd number that is less than 5.

$\forall y\ (\text{Even}(y) \wedge \text{Odd}(y))$

All numbers are both even and odd.

Translations

More practice in section and on homework.

Also a reading on the webpage –

- An explanation of why "for any" is not a great way to translate $\forall$ (even though it looks like a good option on the surface)
- More information on what happens with multiple quantifiers (we'll discuss more on Wednesday).

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \to \texttt{Equal}(x, 2))$

Is this true?

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

Is this true?

TRICK QUESTION! It depends on the domain.

Prime Numbers	Positive Integers	Odd integers
True	False	True (vacuously)

One Technical Matter

How do we parse sentences with quantifiers?

What's the "order of operations?"

We will usually put parentheses right after the quantifier and variable to make it clear what's included. If we don't, it's the rest of the expression.

Be careful with repeated variables…they don't always mean what you think they mean.

$\forall x\big(P(x)\big) \wedge \forall x(Q(x))$ are different x's.

Bound Variables

What happens if we repeat a variable?

Whenever you introduce a new quantifier with an already existing variable, it "takes over" that name until its expression ends.

$$\forall x(P(x) \wedge \forall x[Q(x)] \wedge R(x))$$

It's common (albeit somewhat confusing) practice to reuse a variables when it "wouldn't matter".

Never do something like the above: where a single name switches from gold to purple back to gold. Switching from gold to purple only is usually fine…but names are cheap.

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

For every fruit, if it is not ripe then it is not tasty.

There is a fruit that is sliced and diced.

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

$$\exists x(\mathtt{Tasty}(x) \wedge \mathtt{Ripe}(x))$$

For every fruit, if it is not ripe then it is not tasty.

$$\forall x(\neg\mathtt{Ripe}(x) \rightarrow \neg\mathtt{Tasty}(x))$$

There is a fruit that is sliced and diced.

$$\exists x(\mathtt{Sliced}(x) \wedge \mathtt{Diced}(x))$$

Domain Restriction

Quantifiers

$\forall$ (for **A**ll) and $\exists$ (there **E**xists)

Write these statements in predicate logic with quantifiers. Let your domain of discourse be "cats"

This sentence implicitly makes a statement about all cats!

If a cat is fat, then it is happy.

$$\forall x[\texttt{Fat}(x) \rightarrow \texttt{Happy}(x)]$$

Quantifiers

Writing implications can be tricky when we change the domain of discourse.

For every cat: if the cat is fat, then it is happy.

Domain of Discourse: cats $\forall x[\texttt{Fat}(x) \rightarrow \texttt{Happy}(x)]$

What if we change our domain of discourse to be all mammals?
We need to limit x to be a cat. How do we do that?

$\forall x[(\texttt{Cat}(x) \wedge \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$ $\forall x[\texttt{Cat}(x) \wedge (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

Quantifiers

Which of these translates "For every cat: if a cat is fat then it is happy."
when our domain of discourse is "mammals"?

$\forall x[(\texttt{Cat}(x) \land \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$

For all mammals, if x is a cat and fat then it is happy
[if x is not a cat, the claim is vacuously true, you can't use the promise for anything]

$\forall x[\texttt{Cat}(x) \land (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

For all mammals, that mammal is a cat and if it is fat then it is happy.
[what if x is a dog? Dogs are in the domain, but…uh-oh. This isn't what we meant.]

To "limit" variables to a portion of your domain of discourse under a universal quantifier add a hypothesis to an implication.

Quantifiers

Existential quantifiers need a different rule:

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

There is a dog who is not happy.

Domain of discourse: dogs

$\exists x(\neg \texttt{Happy}(x))$

Quantifiers

Which of these translates “There is a dog who is not happy.”
when our domain of discourse is “mammals”?

$\exists x[\texttt{Dog}(x) \rightarrow \neg\texttt{Happy}(x))]$

There is a mammal, such that if x is a dog then it is not happy.
[this can’t be right – plug in a cat for x and the implication is true]

$\exists x[(\texttt{Dog}(x) \wedge \neg\texttt{Happy}(x))]$

There is a mammal that is both a dog and not happy.
[this one is correct!]

To “limit” variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

Why are the rules what they are?

A universal quantifier is a "Big AND"

For a domain of discourse of $\{e_1, e_2, \dots, e_k\}$

$\forall x(P(x))$ means $P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k)$

Now let's say our domain is $\{e_1, e_2, \dots, e_k, f_1, f_2, \dots, f_j\}$ where f_i are the irrelevant parts of the bigger domain (non-cat-mammals). We want the expression to be

$P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k) \wedge T \wedge T \dots \wedge T$

$\forall x(RightSubDomain(x) \rightarrow P(x))$ does that!

Why are the rules what they are?

An existential quantifier is a "Big OR"

For a domain of discourse of $\{e_1, e_2, \ldots, e_k\}$

$\exists x(P(x))$ means $P(e_1) \vee P(e_2) \vee \cdots \vee P(e_k)$

Now let's say our domain is $\{e_1, e_2, \ldots, e_k, f_1, f_2, \ldots, f_j\}$ where f_i are the irrelevant parts of the bigger domain (non-cat-mammals). We want the expression to be

$P(e_1) \vee P(e_2) \vee \cdots \vee P(e_k) \vee F \vee F \ldots \vee F$

$\exists x(RightSubDomain(x) \wedge P(x))$ does that!

Negating Quantifiers

What happens when we negate an expression with quantifiers?

What does your intuition say?

Original	Negation
Every positive integer is prime	There is a positive integer that is not prime.
$\forall x \texttt{ Prime}(x)$ Domain of discourse: positive integers	$\exists x(\neg \texttt{ Prime}(x))$ Domain of discourse: positive integers

Negating Quantifiers

Let's try on an existential quantifier…

Original	Negation
There is a positive integer which is prime and even.	Every positive integer is composite or odd.
$\exists x(\texttt{Prime}(x) \wedge \texttt{Even}(x))$ Domain of discourse: positive integers	$\forall x(\neg\texttt{Prime}(x) \vee \neg\texttt{Even}(x))$ Domain of discourse: positive integers

To negate an expression with a quantifier

1. Switch the quantifier ($\forall$ becomes $\exists$, $\exists$ becomes $\forall$)
2. Negate the expression inside

Negation

Translate these sentences to predicate logic, then negate them.

All cats have nine lives.

$$\forall x\big(Cat(x) \rightarrow NumLives(x, 9)\big)$$

$\exists x(Cat(x) \wedge \neg\big(NumLives(x, 9)\big))$ "There is a cat without 9 lives.

All dogs love every person.

$$\forall x \forall y\big(Dog(x) \wedge Human(y) \rightarrow Love(x, y)\big)$$

$\exists x \exists y(Dog(x) \wedge Human(y) \wedge \neg Love(x, y))$ "There is a dog who does not love someone." "There is a dog and a person such that the dog doesn't love that person."

There is a cat that loves someone.

$$\exists x \exists y(Cat(x) \wedge Human(y) \wedge Love(x, y)$$

$$\forall x \forall y(Cat(x) \wedge Human(y) \rightarrow \neg Love(x, y))$$

"For every cat and every human, the cat does not love that human."
"Every cat does not love any human" ("no cat loves any human")

Negation with Domain Restriction

$\exists x \exists y(Cat(x) \land Human(y) \land Love(x, y))$

$\forall x \forall y([Cat(x) \land Human(y)] \rightarrow \neg Love(x, y))$

There are lots of equivalent expressions to the second. This one is by far the best because it reflects the domain restriction happening. How did we get there?

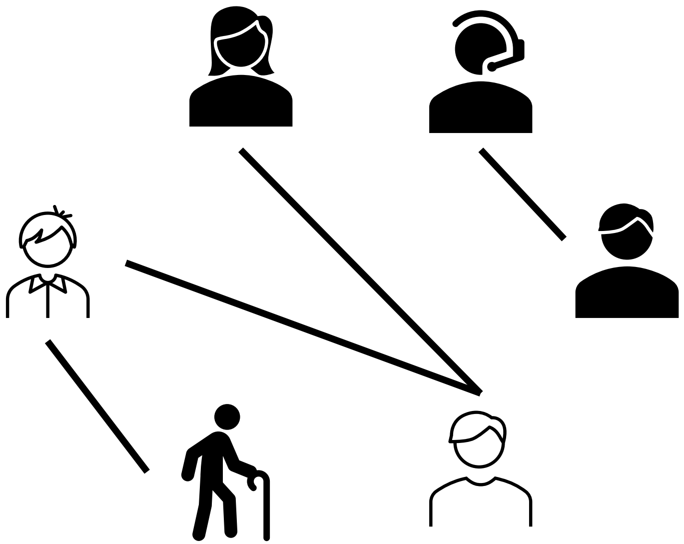
There's a problem in this week's section handout showing similar algebra.

Nested Quantifiers

Nested Quantifiers

Translate these sentences using only quantifiers and the predicate `AreFriends`(x, y)

Everyone is friends with someone.



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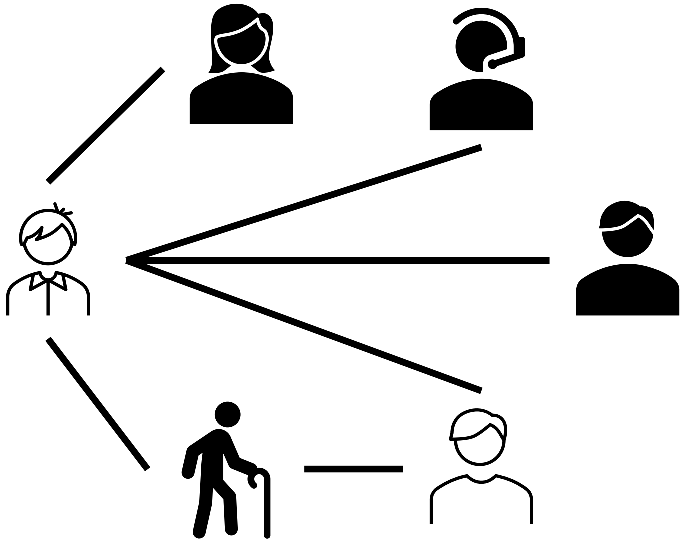

Nested Quantifiers

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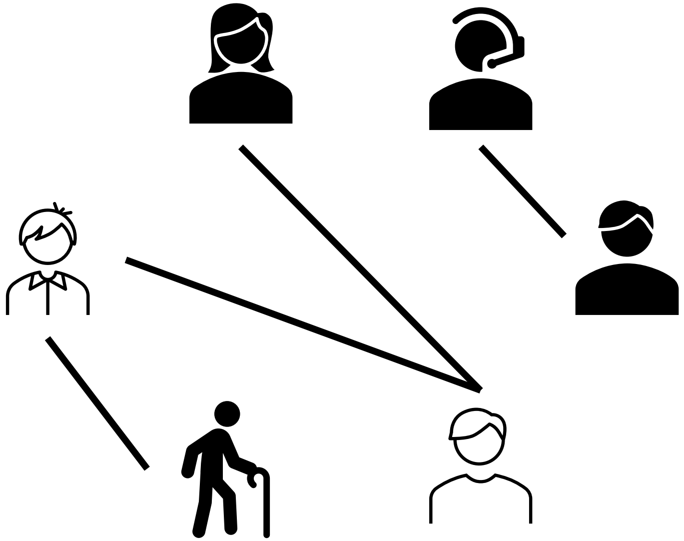
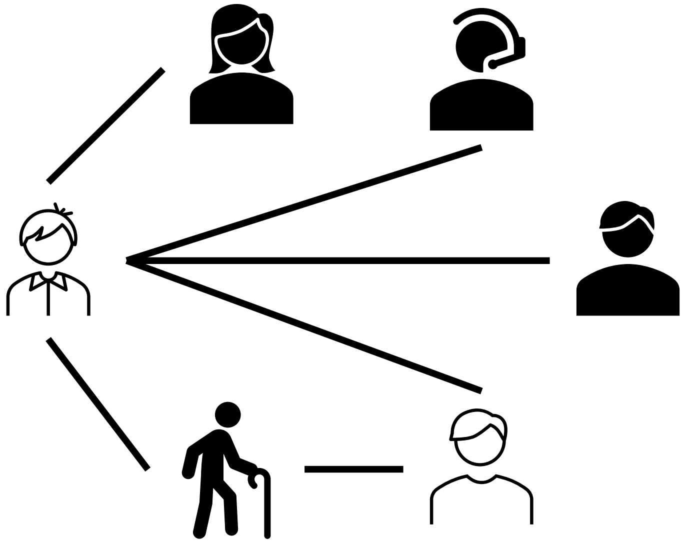


$\forall x(\exists y \texttt{ AreFriends}(x, y))$

$\forall x \exists y \texttt{ AreFriends}(x, y)$

Someone is friends with everyone.



$\exists x(\forall y \texttt{ AreFriends}(x, y))$

$\exists x \forall y \texttt{ AreFriends}(x, y)$

Nested Quantifiers

$\forall x \exists y \, P(x, y)$

"For every x there exists a y such that $P(x, y)$ is true."

y might change depending on the x (people have different friends!).

$\exists x \forall y \, P(x, y)$

"There is an x such that for all $y, P(x, y)$ is true."

There's a special, magical x value so that $P(x, y)$ is true regardless of y.

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

$\forall x \exists y P(x, y)$

y

x

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Nested Quantifiers

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And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

A row, where every entry is T

$\forall x \exists y P(x, y)$

In every row there must be a T

y

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

x

Keep everything in order

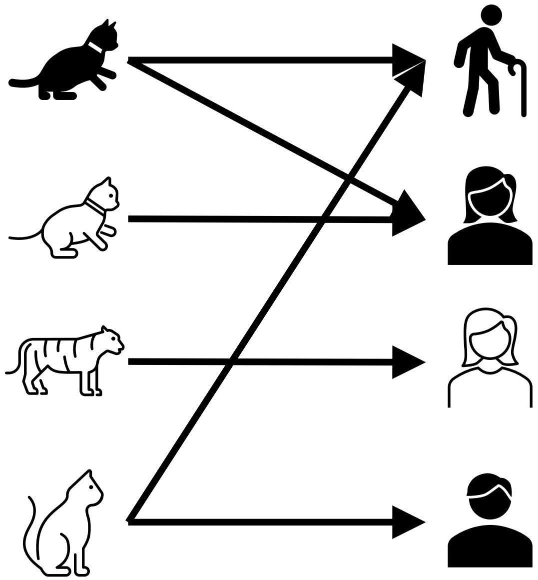
Keep the quantifiers in the same order in English as they are in the logical notation.

“There is someone out there for everyone” is a $\forall x \exists y$ statement in “everyday” English.

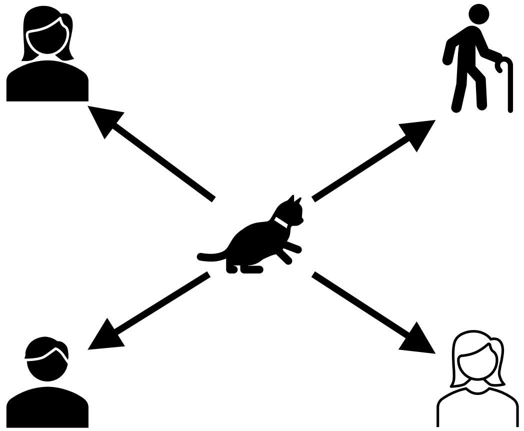
It would **never** be phrased that way in “mathematical English” We’ll only every write “for every person, there is someone out there for them.”

Try it yourselves

Every cat loves some human.



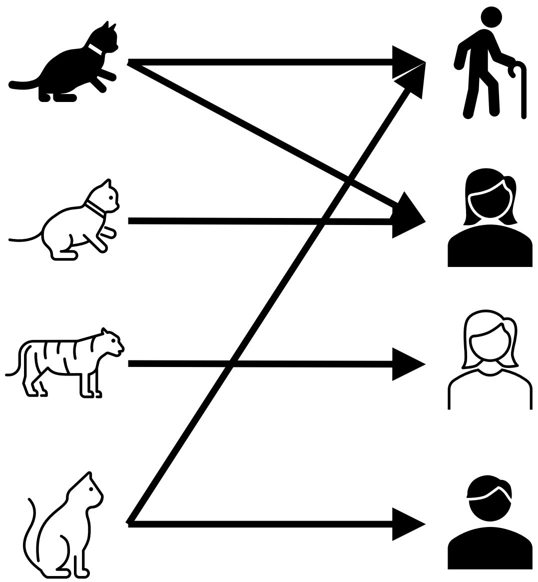
There is a cat that loves every human.



Let your domain of discourse be mammals.
Use the predicates `Cat`(x), `Dog`(x), and `Loves`(x, y) to mean x loves y.

Try it yourselves

Every cat loves some human.



$\forall x\ (\mathtt{Cat}(x) \rightarrow \exists y[\mathtt{Human}(y) \wedge \mathtt{Loves}(x, y)])$

$\forall x \exists y(\mathtt{Cat}(x) \rightarrow [\mathtt{Human}(y) \wedge \mathtt{Loves}(x, y)])$

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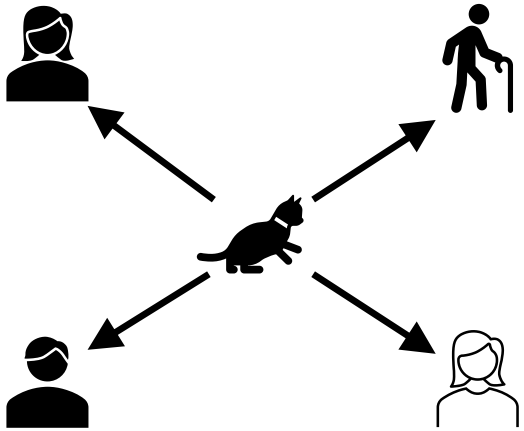

$\exists x\ (\mathtt{Cat}(x) \wedge \forall y[\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x, y)])$

$\exists x \forall y(\mathtt{Cat}(x) \wedge [\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x, y)])$

Negation

How do we negate nested quantifiers?

The old rule still applies.

To negate an expression with a quantifier
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

$$\neg(\forall x \exists y \forall z\, [P(x,y) \wedge Q(y,z)])$$

$$\exists x(\neg(\exists y \forall z\, [P(x,y) \wedge Q(y,z)]))$$

$$\exists x \forall y(\neg(\forall z[P(x,y) \wedge Q(y,z)]))$$

$$\exists x \forall y \exists z(\neg[P(x,y) \wedge Q(y,z)])$$

$$\exists x \forall y \exists z[\neg P(x,y) \vee \neg Q(y,z)]$$

More Translation

For each of the following, translate it, then say whether the statement is true. Let your domain of discourse be integers.

For every integer, there is a greater integer.

$\forall x \exists y(\texttt{Greater}(y, x))$ (This statement is true: y can be $x + 1$ [y depends on x])

There is an integer x, such that for all integers y, xy is equal to 1.

$\exists x \forall y(\texttt{Equal}(xy, 1))$ (This statement is false: no single value of x can play that role for every y.)

$\forall y \exists x(\texttt{Equal}(x + y, 1))$

For every integer, y, there is an integer x such that $x + y = 1$
(This statement is true, y can depend on x)