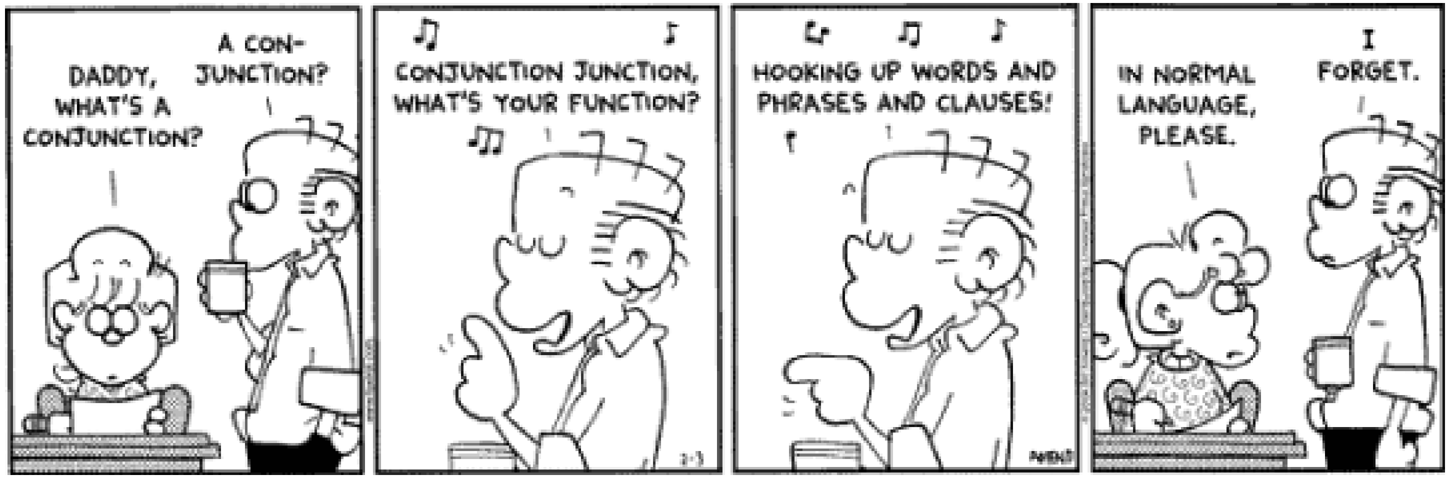

More Logic, Equivalences, Symbolic Proofs

CSE 311 Autumn 2023
Lecture 2

Announcements

Lecture recordings on “panopto” – link is posted on ed.

HW1 comes out tonight – on the assignments tab on the webpage

Due Friday evening.

You’ll submit to gradescope.

OH have started!

mix of zoom + in-person

Since no HW is out yet, not too many today. Lots on Monday.

Visit early! We get very busy right before deadlines!

CC1 and 2 will both be due Monday morning (delayed deadline for CC1)

Still make sure you can get on gradescope right away!

Consider enrolling in 390Z

There's still space!

More info on the 390z course webpage

You need an add code, the form to get one is on the course web.

Today

More on implications

Simplification Rules

Our first proof (maybe)!

Implication ($p \to q$)

p q

"If it's raining, then I have my umbrella"

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

It's useful to think of implications as promises. An implication is false exactly when you can ***demonstrate*** *I'm lying.*

	It's raining	It's not raining
I have my umbrella	No lie. True	No lie. True
I do not have my umbrella	**LIE! False**	No lie. True

$p \rightarrow q$

$p \rightarrow q$ and $q \rightarrow p$ are different implications!

"If the sun is out, then we have class outside."

"If we have class outside, then the sun is out."

Only the first is useful to you when you see the sun come out.

Only the second is useful if you forgot your umbrella.

$p \rightarrow q$

Implication:

p implies q

whenever p is true q must be true

if p then q

q if p

p is sufficient for q

p only if q

q is necessary for p

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

a if b

$b \rightarrow a$

Implications are super useful, so there are LOTS of translations.
You'll learn these in detail in section.

More Connectives

A More Complicated Statement

"Robbie knows the Pythagorean Theorem if he is a mathematician and took geometry, and he is a mathematician or did not take geometry."

Is this a proposition?

We'd like to *understand* what this proposition means.

In particular, is it true?

A Compound Proposition

"Robbie knows the Pythagorean Theorem if he is a mathematician and took geometry, and he is a mathematician or did not take geometry."

We'd like to *understand* what this proposition means.

First find the simplest (**atomic) propositions**:

p	"Robbie knows the Pythagorean Theorem"
q	"Robbie is a mathematician"
r	"Robbie took geometry"

(p if (q and r)) and (q or (not r))

$$(p \text{ if } (q \wedge r)) \wedge (q \vee (\neg r))$$

Parentheses…

"Robbie knows the Pythagorean Theorem if he is a mathematician and took geometry, and he is a mathematician or did not take geometry."

$$(p \text{ if } (q \wedge r)) \wedge (q \vee (\neg r))$$

p	**"Robbie knows the Pythagorean Theorem"**
q	**"Robbie is a mathematician"**
r	**"Robbie took geometry"**

How did we know where to put the parentheses?

- Subtle English grammar choices (top-level parentheses are independent clauses).
- Context/which parsing will make more sense.
- Conventions

A reading on this is coming soon!

Back to the Compound Proposition…

p if $(q \wedge r)$

"Robbie knows the Pythagorean Theorem if he is a mathematician and took geometry, and he is a mathematician or did not take geometry."

$(p \text{ if } (q \wedge r)) \wedge (q \vee (\neg r))$

p	**"Robbie knows the Pythagorean Theorem"**
q	**"Robbie is a mathematician"**
r	**"Robbie took geometry"**

T F

What promise am I making?

$((q \wedge r) \rightarrow p) \wedge (q \vee (\neg r))$ $\qquad$ $(p \rightarrow (q \wedge r)) \wedge (q \vee (\neg r))$

The first one! Being a mathematician and taking geometry is the condition. Knowing the Pythagorean Theorem is the promise.

Analyzing the Sentence with a Truth Table

p	q	r	$\neg r$	$q \vee \neg r$	$q \wedge r$	$(q \wedge r) \rightarrow p$	$((q \wedge r) \rightarrow p) \wedge (q \vee \neg r)$
F	F	F	T	T	F	T	T
F	F	T	F	F	F	T	F
F	T	F	T	T	F	T	T
F	T	T	F	T	T	F	F
T	F	F	T	T	F	T	T
T	F	T	F	F	F	T	F
T	T	F	T	T	F	T	T
T	T	T	F	T	T	T	T

Order of Operations

Just like you were taught PEMDAS

e.g. $3 + 2 \cdot 4 = 11$ not 24.

Logic also has order of operations.

Parentheses

Negation

And

Or, exclusive or

Implication

Biconditional

For this class: each of these is it's own level!
e.g. "and"s have precedence over "or"s

Within a level, apply from left to right.

Other authors place And, Or at the same level – it's good practice to use parentheses even if not required.

Logical Connectives

Negation (not)	$\neg p$
Conjunction (and)	$p \wedge q$
Disjunction (or)	$p \vee q$
Exclusive Or	$p \oplus q$
Implication(if-then)	$p \longrightarrow q$
Biconditional	$p \longleftrightarrow q$

These ideas have been around for so long most have at least two names.

Two more connectives to discuss!

Biconditional: $p \leftrightarrow q$

Think: $(p \rightarrow q) \wedge (q \rightarrow p)$

p if and only if *q*

p iff q

p is equivalent to *q*

p implies *q* and *q* implies *p*

p is necessary and sufficient for *q*

p	q	$p \leftrightarrow q$

Biconditional: $p \leftrightarrow q$

Think: $(p \rightarrow q) \wedge (q \rightarrow p)$

p if and only if *q*

p iff q

p is equivalent to *q*

p implies *q* and *q* implies *p*

p is necessary and sufficient for *q*

p	q	$p \leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

$p \leftrightarrow q$ is the proposition: "p and q have the same truth value."

Exclusive Or

Exactly one of the two is true.

$p \oplus q$

p	q	$p \oplus q$

In English "either p or q" is the most common, but be careful.

Often translated "p or q" where you're just supposed to understand that exclusive or is meant (instead of the normal inclusive or).

Try to say "either…or…" in your own writing.

Exclusive Or

Exactly one of the two is true.

$p \oplus q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

In English "either p or q" is the most common, but be careful.

Often translated "p or q" where you're just supposed to understand that exclusive or is meant (instead of the normal inclusive or).

Try to say "either…or…" in your own writing.

Today

A proof!

We want to be able to make iron-clad guarantees that something is true.

And convince others that we really have ironclad guarantees.

But first, some notation.

Logical Equivalence

We will want to talk about whether two propositions are "the same."

Two propositions are "equal" ($=$) if they are character-for-character identical.

$p \wedge q = p \wedge q$ but $p \wedge q \neq q \wedge p$

We almost never ask whether propositions are equal. It's not an interesting question.

Two propositions are "equivalent" ($\equiv$) if they always have the same truth value.

$p \wedge q \equiv p \wedge q$ and $p \wedge q \equiv q \wedge p$

But $p \wedge q \not\equiv p \vee q$
When p is true and q is false: $p \wedge q$ is false, but $p \vee q$ is true.

$A \leftrightarrow B$ vs. $A \equiv B$

$A \equiv B$ is an ***assertion over all possible truth values*** that A and B always have the same truth values.

Use $A \equiv B$ when you're manipulating propositions ("doing algebra")

$A \leftrightarrow B$ is a ***proposition*** that may be true or false depending on the truth values of the variables in A and B.

This distinction will be easier to understand after you see us use them both a few time.

Simplification and Proofs

Manipulating Expressions

When we're doing algebra, we can apply rules to transform expressions

$(a+b)(c+d) = ac + ad + bc + bd$ or $ab + ac = a(b+c)$

We want rules for logical expressions too.

For two rules, we'll:

1. Derive it/make sure we understand why it's true.
2. Practice using it.

By the end of the course, you'll do these "automatically" on full sentences; for now we'll practice mechanically on symbolic forms.

As you're practicing, don't lose sight of the *intuition* for what you're doing.

De Morgan's Laws

Negate the statement

"my code compiles or there is a bug."

i.e. find a natural English sentence that says $\neg(p \vee q)$

"the following is not true: my code compiles or there is a bug"

Hint: when it the original sentence false?

'or' means 'at least one is true' so to negate, we need to say 'neither is true' or equivalently 'both are false'

$\neg p \wedge \neg q$

"my code does not compile and there is not a bug"

De Morgan’s Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

is a general rule. It’s always true for any propositions p and q.
This is one of De Morgan’s Laws.

The other is:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

De Morgan's Laws

Example: $\neg(p \land q) \equiv \neg p \lor \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \lor \neg q$	$p \land q$	$\neg(p \land q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Every line is the same!
So these expressions are equivalent.

De Morgan’s Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

```
if (!(front != null && value > front.data))
      front = new ListNode(value, front);
else {
      ListNode current = front;
      while (current.next != null && current.next.data < value))
           current = current.next;
      current.next = new ListNode(value, current.next);
}
```

De Morgan's Laws

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

```
!(front != null && value > front.data)
```

$$\equiv$$

```
front == null || value <= front.data
```

You've been using these for a while!

Law of Implication

Implications are hard.
AND/OR/NOT make more intuitive sense to me…
can we rewrite implications using just ANDs ORs and NOTs?

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

One approach: think "when is this implication false?" then negate it (you might want one of DeMorgan's Laws!

Law of Implication

Implications are hard.
AND/OR/NOT make more intuitive sense to me…
can we rewrite implications using just ANDs ORs and NOTs?

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Seems like we might want $\neg(p \wedge \neg q)$

$\neg p \vee q$

Seems like a reasonable guess.
So is it true? Is $\neg p \vee q \equiv p \to q$?

Law of Implication

$$\neg p \vee q \equiv p \rightarrow q$$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \rightarrow q}$	$\boldsymbol{\neg p}$	$\boldsymbol{\neg p \vee q}$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Every line is the same!
So these expressions are equivalent.

Properties of Logical Connectives

We’ve derived two facts about logical connectives.

There’s a lot more. A LOT more.

The next slide is a list of a bunch of them…

- Most of these are much less complicated than the last two, so we won’t go through them in detail.
- DO NOT freak out about how many there are. We will always provide you the list on the next slide (no need to memorize).

Properties of Logical Connectives

You don't have to memorize this list!

These identities hold for all propositions p, q, r

- Identity
  - $p \land \mathrm{T} \equiv p$
  - $p \lor \mathrm{F} \equiv p$
- Domination
  - $p \lor \mathrm{T} \equiv \mathrm{T}$
  - $p \land \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \lor p \equiv p$
  - $p \land p \equiv p$
- Commutative
  - $p \lor q \equiv q \lor p$
  - $p \land q \equiv q \land p$
- Associative
  - $(p \lor q) \lor r \equiv r \lor (q \lor r)$
  - $(p \land q) \land r \equiv p \land (q \land r)$
- Distributive
  - $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
  - $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
- Absorption
  - $p \lor (p \land q) \equiv p$
  - $p \land (p \lor q) \equiv p$
- Negation
  - $p \lor \neg p \equiv \mathrm{T}$
  - $p \land \neg p \equiv \mathrm{F}$

Using Our Rules

WOW that was a lot of rules.

Why do we need them? Simplification!

Let's go back to the "law of implication" example.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

When is the implication true? Just "or" each of the three "true" lines!

$$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$$

Also seems pretty reasonable

So is $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv (\neg p \vee q)$?

i.e. are these both alternative representations of $p \to q$?

Our First Proof

We could make another truth table (you should! It's a good exercise)

But we have another technique that is nicer.

Let's try that one

Then talk about why it's another good option.

We're going to give an iron-clad guarantee that:

$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv \neg p \vee q$

i.e. that this is another valid "law of implication"

Our First Proof

This will be a long proof! Longer than most of the ones on homeworks.

I'm starting with a hard one so you see all the tricks.

This process will be easier if we change variables, we're going to show

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv \neg a \vee b$

How do we write a proof?

It's not always plug-and-chug…we'll be highlighting strategies throughout the quarter.

To start with:

Make sure we know what we want to show…

Our First Proof

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv$

None of the rules look like this

Practice of Proof-Writing:
Big Picture…WHY do we think this might be true?

The last two "pieces" came from the vacuous proof lines…maybe the "$\neg a$" came from there? Maybe that **simplifies** down to $\neg a$

$\equiv (\neg a \vee b)$

Let’s apply a rule

$(\neg a \wedge b) \vee (\neg a \wedge \neg b)$

The law says:

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$(\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv \neg p \wedge (b \vee \neg b)$

Our First Proof

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv$

None of the rules look like this

Practice of Proof-Writing:
Big Picture...WHY do we think this might be true?

The last two "pieces" came from the vacuous proof lines...maybe the "$\neg a$" $\equiv (\neg a \vee b)$ came from there? Maybe that **simplifies** down to $\neg a$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Set ourselves an intermediate goal.
Let's try to simplify those last two pieces

Associative law
Connect up the things we're working on.

$$\equiv (\neg a \vee b)$$

Our First Proof

$$
\begin{aligned}
(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) &\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)] \\
&\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]
\end{aligned}
$$

Set ourselves an intermediate goal.
Let's try to simplify those last two pieces

Distributive law
We think $\neg a$ is important, let's isolate it.

$$\equiv (\neg a \vee b)$$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$$

Set ourselves an intermediate goal.
Let's try to simplify those last two pieces

Negation
Should make things simpler.

$$\equiv (\neg a \vee b)$$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$$
$$\equiv (a \wedge b) \vee [\neg a]$$

Set ourselves an intermediate goal.
Let's try to simplify those last two pieces

Identity
Should make things simpler.

$$\equiv (\neg a \vee b)$$

Our First Proof

$$
\begin{aligned}
(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) &\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)] \\
&\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)] \\
&\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}] \\
&\equiv (a \wedge b) \vee [\neg a]
\end{aligned}
$$

Stay on target:
We met our intermediate goal.
Don't forget the final goal!
We want to end up at $(\neg a \vee b)$

If we apply the distribution rule,
We'd get a $(\neg a \vee b)$

$$\equiv (\neg a \vee b)$$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Stay on target:
We met our intermediate goal.
Don‘t forget the final goal!
We want to end up at $(\neg a \vee b)$

$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$$
$$\equiv (a \wedge b) \vee [\neg a]$$
$$\equiv [\neg a] \vee (a \wedge b)$$

If we apply the distribution rule,
We’d get a $(\neg a \vee b)$

Commutative
Make the expression look exactly like the law (more on this later)

$$\equiv (\neg a \vee b)$$

Our First Proof

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$

$\equiv (a \wedge b) \vee [\neg a]$

$\equiv [\neg a] \vee (a \wedge b)$

$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$

Stay on target:
We met our intermediate goal.
Don't forget the final goal!
We want to end up at $(\neg a \vee b)$

If we apply the distribution rule,
We'd get a $(\neg a \vee b)$

Distributive
Creates the $(\neg a \vee b)$ we were hoping for.

$\equiv (\neg a \vee b)$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$

$$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$$

$$\equiv (a \wedge b) \vee [\neg a]$$

$$\equiv [\neg a] \vee (a \wedge b)$$

$$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$$

$$\equiv (a \vee \neg a) \wedge (\neg a \vee b)$$

$$\equiv \mathrm{T} \wedge (\neg a \vee b)$$

$$\equiv (\neg a \vee b)$$

Stay on target:
We met our intermediate goal.
Don't forget the final goal!
We want to end up at $(\neg a \vee b)$

If we apply the distribution rule,
We'd get a $(\neg a \vee b)$

Commutative
Make the expression look exactly like the law (more on this later)

Negation
Simplifies the part we want to disappear.

Simplify $\mathtt{T} \land (\neg a \lor b)$ to $(\neg a \lor b)$

These identities hold for all propositions p, q, r

- Identity
  - $p \land \mathrm{T} \equiv p$
  - $p \lor \mathrm{F} \equiv p$
- Domination
  - $p \lor \mathrm{T} \equiv \mathrm{T}$
  - $p \land \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \lor p \equiv p$
  - $p \land p \equiv p$
- Commutative
  - $p \lor q \equiv q \lor p$
  - $p \land q \equiv q \land p$
- Associative
  - $(p \lor q) \lor r \equiv r \lor (q \lor r)$
  - $(p \land q) \land r \equiv p \land (q \land r)$
- Distributive
  - $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
  - $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
- Absorption
  - $p \lor (p \land q) \equiv p$
  - $p \land (p \lor q) \equiv p$
- Negation
  - $p \lor \neg p \equiv \mathrm{T}$
  - $p \land \neg p \equiv \mathrm{F}$

Our First Proof

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b)$

$\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$

$\equiv (a \wedge b) \vee [\neg a]$

$\equiv [\neg a] \vee (a \wedge b)$

$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$

$\equiv (a \vee \neg a) \wedge (\neg a \vee b)$

$\equiv \mathrm{T} \wedge (\neg a \vee b)$

$\equiv (\neg a \vee b) \wedge \mathrm{T}$

$\equiv (\neg a \vee b)$

Stay on target:
We met our intermediate goal.
Don't forget the final goal!
We want to end up at $(\neg a \vee b)$

If we apply the distribution rule,
We'd get a $(\neg a \vee b)$

Commutative followed by **Identity**
Look exactly like the law, then apply it.

We're done!!!

Commutativity

We had the expression $(a \wedge b) \vee [\neg a]$

But before we applied the distributive law, we switched the order…why?

The law says $p \vee (q \wedge r) \equiv (\mathrm{p} \vee q) \wedge (p \vee r)$

not $(q \wedge r) \vee p \equiv (q \vee p) \wedge (r \vee p)$

So **technically** we needed to commute first.

Eventually (in about 2 weeks) we'll skip this step. For now, we're doing two separate steps.

Remember this is the "training wheel" stage. The point is to be careful.

More on Our First Proof

We now have an ironclad guarantee that

$(a \land b) \lor (\neg a \land b) \lor (\neg a \land \neg b) \equiv (\neg a \lor b)$

Hooray! But we could have just made a truth-table. Why a proof?

Here's one reason.

Proofs don't *just* give us an ironclad guarantee. They're also an explanation of *why* the claim is true.

The key insight to our simplification was "the last two pieces were the vacuous truth parts – the parts where p was false"

That's in there, *in the proof.*

Our First Proof

$$
\begin{aligned}
(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) &\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)] && \text{Associative} \\
&\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)] && \text{Distributive} \\
&\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}] && \text{Negation} \\
&\equiv (a \wedge b) \vee [\neg a] && \text{Identity} \\
&\equiv [\neg a] \vee (a \wedge b) && \text{Commutative} \\
&\equiv (\neg a \vee a) \wedge (\neg a \vee b) && \text{Distributive} \\
&\equiv (a \vee \neg a) \wedge (\neg a \vee b) && \text{Commutative} \\
&\equiv \mathrm{T} \wedge (\neg a \vee b) && \text{Negation} \\
&\equiv (\neg a \vee b) \wedge \mathrm{T} && \text{Commutative} \\
&\equiv (\neg a \vee b) && \text{Identity}
\end{aligned}
$$

The last two terms are "vacuous truth" – they simplify to $\neg a$

a no longer matters in $a \wedge b$ if $\neg a$ automatically makes the expression true.

More on Our First Proof

With practice (and quite a bit of squinting) you can see not just the ironclad guarantee, but also the reason why something is true.

That’s not easy with a truth table.

Proofs can also communicate intuition about why a statement is true.

We’ll practice extracting intuition from proofs more this quarter.

Take the conjunction of the following two propostions:

If Robbie is a mathematician and took geometry, then he knows the Pythagorean theorem

Robbie is a mathematicaion or did not take geometry.