Section 10: Irregularity, Cardinality & Uncomputability

1. Irregularity

(a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n 1^n 0^n \ : \ n \geq 0\}$ is not regular.

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n (12)^m \ : \ n \geq m \geq 0\}$ is not regular.

2. Cardinality

(a) You are a pirate. You begin in a square on a 2D grid which is infinite in all directions. In other words, wherever you are, you may move up, down, left, or right. Some single square on the infinite grid has treasure on it. Find a way to ensure you find the treasure in finitely many moves.

(b) Prove that $\{3x \ : \ x \in \mathbb{N}\}$ is countable.

(c) Prove that the set of irrational numbers is uncountable.
Hint: Use the fact that the rationals are countable and that the reals are uncountable.

(d) Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

3. Countable Unions

(a) Show that $\mathbb{N} \times \mathbb{N}$ is countable.

Hint: How did we show the rationals were countable?

(b) Show that the countable union of countable sets is countable. That is, given a collection of sets $S_1, S_2, S_2, \ldots$ such that S_i is countable for all $i \in \mathbb{N}$, show that

$$S = S_1 \cup S_2 \cup \cdots = \{x : x \in S_i \text{ for some } i\}$$

is countable.

Hint: Find a way labeling the elements and see if you can apply the previous part to construct an onto function from $\mathbb{N}$ to S.

4. Uncomputability

(a) Let $\Sigma = \{0, 1\}$. Prove that the set of palindromes is decidable.

(b) Prove that the set $\{(\texttt{CODE}(R), x, y) \ : \ R$ is a program and $R(x) \neq R(y)\}$ is undecidable where $R(x)$ is the output string that R produces on input x if R halts and we write $R(x) = \uparrow$ if R runs forever.

5. Functions

Let $f : X \to Y$ be a function. For a subset C of X, define $f(C)$ to be the set of elements that f sends C to. In other words, $f(C) = \{f(c) : c \in C\}$.

Let A, B be subsets of X. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.