Section 09: Solutions

1. CFGs

Write a context-free grammar to match each of these languages.

(a) All binary strings that start with 11.
Solution:

$$\mathbf{S} \to 11\mathbf{T}$$
$$\mathbf{T} \to 1\mathbf{T} \mid 0\mathbf{T} \mid \varepsilon$$

(b) All binary strings that contain at most one 1.
Solution:

$$\mathbf{S} \to \mathbf{ABA}$$
$$\mathbf{A} \to 0\mathbf{A} \mid \varepsilon$$
$$\mathbf{B} \to 1 \mid \varepsilon$$

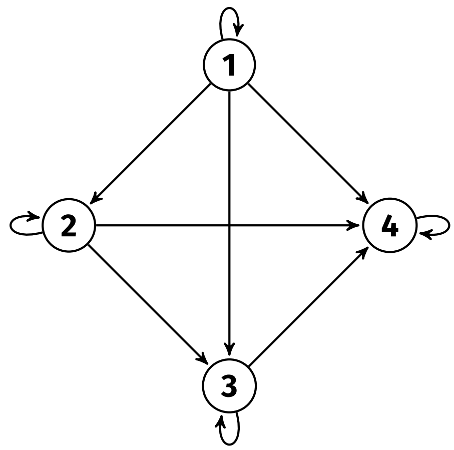
(c) All strings over 0, 1, 2 with the same number of 1s and 0s and exactly one 2.
Hint: Try modifying the grammar from Section 8 2c for binary strings with the same number of 1s and 0s (You may need to introduce new variables in the process).
Solution:

$$\mathbf{S} \to 2\mathbf{T} \mid \mathbf{T}2 \mid \mathbf{ST} \mid \mathbf{TS} \mid 0\mathbf{S}1 \mid 1\mathbf{S}0$$
$$\mathbf{T} \to \mathbf{TT} \mid 0\mathbf{T}1 \mid 1\mathbf{T}0 \mid \varepsilon$$

T is the grammar from Section 8 2c. It generates all binary strings with the same number of 1s and 0s. **S** matches a 2 at the beginning or end. The rest of the string must then match **T** since it cannot have another 2. If neither the first nor last character is a 2, then it falls into the usual cases of matching 0s and 1s, so we can mostly use the same rules as **T**. The main change is that **SS** becomes **ST** | **TS** to ensure that exactly one of the two parts contains a 2. The other change is that there is no ϵ since a 2 must appear somewhere.

2. Relations

(a) Draw the transitive-reflexive closure of $\{(1,2),(2,3),(3,4)\}$.
Solution:



(b) Suppose that R is reflexive. Prove that $R \subseteq R^2$.
Solution:

Suppose $(a, b) \in R$. Since R is reflexive, we know $(b, b) \in R$ as well. Since there is a b such that $(a, b) \in R$ and $(b, b) \in R$, it follows that $(a, b) \in R^2$. Thus, $R \subseteq R^2$.

(c) Consider the relation $R = \{(x, y) \ : \ x = y + 1\}$ on $\mathbb{N}$. Is R reflexive? Transitive? Symmetric? Anti-symmetric?
Solution:

It isn't reflexive, because $1 \neq 1 + 1$; so, $(1, 1) \notin R$. It isn't symmetric, because $(2, 1) \in R$ (because $2 = 1 + 1$), but $(1, 2) \notin R$, because $1 \neq 2 + 1$. It isn't transitive, because note that $(3, 2) \in R$ and $(2, 1) \in R$, but $(3, 1) \notin R$. It is anti-symmetric, because consider $(x, y) \in R$ such that $x \neq y$. Then, $x = y + 1$ by definition of R. However, $(y, x) \notin R$, because $y = x - 1 \neq x + 1$.

(d) Consider the relation $S = \{(x, y) \ : \ x^2 = y^2\}$ on $\mathbb{R}$. Prove that S is reflexive, transitive, and symmetric.
Solution:

Consider $x \in \mathbb{R}$. Note that by definition of equality, $x^2 = x^2$; so, $(x, x) \in S$; so, S is reflexive.

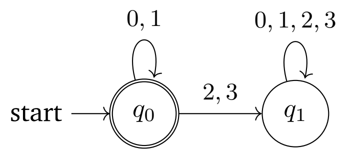
Consider $(x, y) \in S$. Then, $x^2 = y^2$. It follows that $y^2 = x^2$; so, $(y, x) \in S$. So, S is symmetric.

Suppose $(x, y) \in S$ and $(y, z) \in S$. Then, $x^2 = y^2$, and $y^2 = z^2$. Since equality is transitive, $x^2 = z^2$. So, $(x, z) \in S$. So, S is transitive.

3. DFAs, Stage 1

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1, 2, 3\}$.

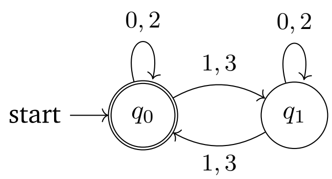
(a) All binary strings.
Solution:



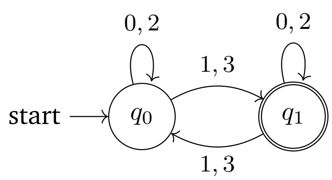
q_0: binary strings

q_1: strings that contain a character which is not 0 or 1.

(b) All strings whose digits sum to an even number.
Solution:



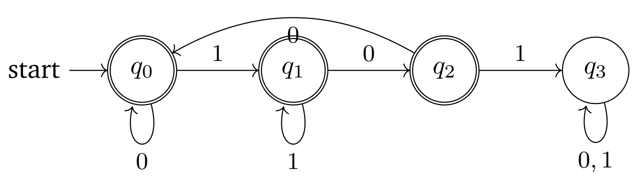
(c) All strings whose digits sum to an odd number.
Solution:



4. DFAs, Stage 2

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1\}$.

(a) All strings which do not contain the substring 101.
Solution:



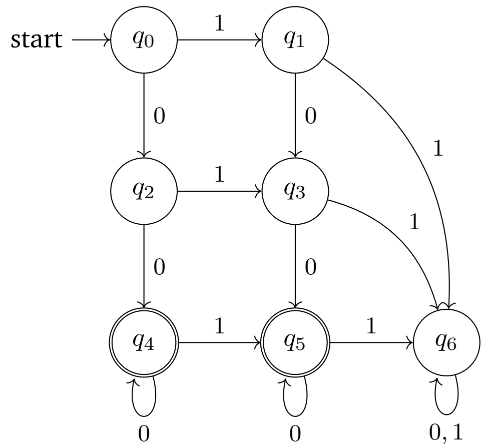
q_3: string that contain 101.

q_2: strings that don't contain 101 and end in 10.

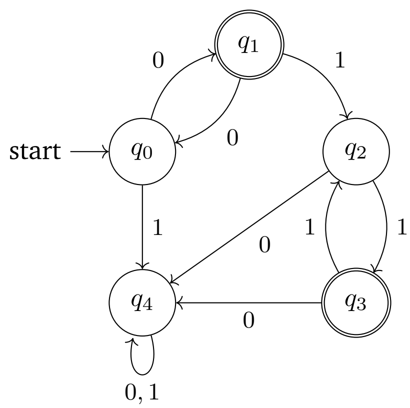
q_1: strings that don't contain 101 and end in 1.

q_0: ε, 0, strings that don't contain 101 and end in 00.

(b) All strings containing at least two 0's and at most one 1.
Solution:

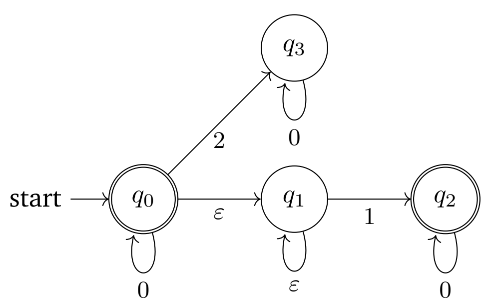


(c) All strings containing an even number of 1's and an odd number of 0's and not containing the substring 10.
Solution:



5. NFAs

(a) What language does the following NFA accept?

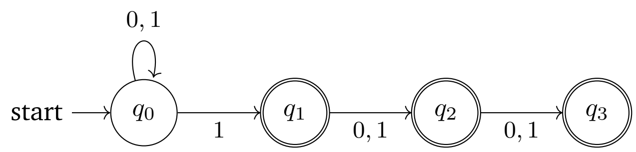


Solution:

All strings of only 0's and 1's not containing more than one 1.

(b) Create an NFA for the language "all binary strings that have a 1 as one of the last three digits".
Solution:

The following is one such NFA:

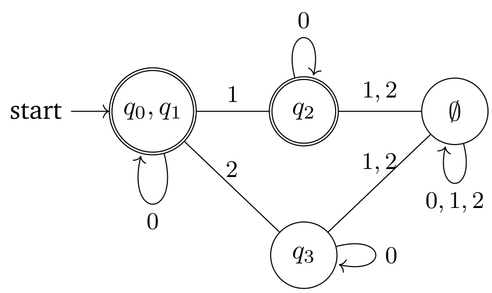


6. DFAs & Minimization

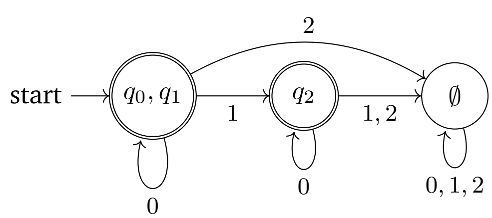
Note: We will not test you on minimization, although you may optionally read the extra slides and do this problem for fun

(a) Convert the NFA from 1a to a DFA, then minimize it.

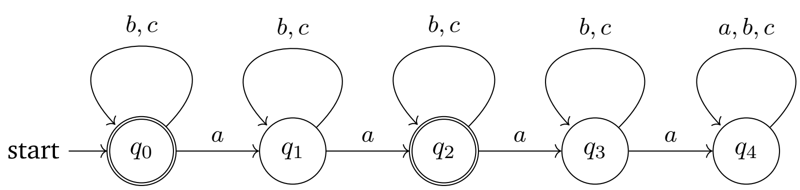
Solution:



Here is the minimized form:



(b) Minimize the following DFA:



Solution:

Step 1: q_0, q_2 are final states and the rest are not final. So, we start with the initial partition with the following groups: group 1 is $\{q_0, q_2\}$ and group 2 is $\{q_1, q_3, q_4\}$.

Step 2: q_1 is sending a to group 1 while q_3, q_4 are sending a to group 2. So, we divide group 2. We get the following groups: group 1 is $\{q_0, q_2\}$, group 3 is $\{q_1\}$ and group 4 is $\{q_3, q_4\}$.

Step 3: q_0 is sending a to group 3 and q_2 is sending a to group 4. So, we divide group 1. We will have the following groups: group 3 is $\{q_1\}$, group 4 is $\{q_3, q_4\}$, group 5 is $\{q_0\}$ and group 6 is $\{q_2\}$.

The minimized DFA is the following: