Final Review Session

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Warm-up: Predicate Logic

Express each of these system specifications using predicates, quantifiers, and logical connectives. For some of these problems, more than one translation will be reasonable depending on your choice of predicates.

- (a) Every user has access to an electronic mailbox
- (b) The system mailbox can be accessed by everyone in the group if the file system is locked.
- (c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.
- (d) At least one router is functioning normally if the throughput is between 100kbps and 500 kbps and the proxy server is not in diagnostic mode.

(a) Every user has access to an electronic mailbox.

Let the domain be users and mailboxes. Let User(x) be "x is a user", let Mailbox(y) be "y is a mailbox", and let Access(x, y) be "x has access to y".

 $\forall x (\text{User}(x) \rightarrow (\exists y (\text{Mailbox}(y) \land \text{Access}(x, y))))$

(b) The system mailbox can be accessed by everyone in the group if the file system is locked.

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(b) The system mailbox can be accessed by everyone in the group if the file system is locked.

Solution 1: Let the domain be people in the group. Let CanAccessSM(*x*) be "*x* has access to the system mailbox". Let *p* be the proposition "the file system is locked."

 $p \rightarrow \forall x \operatorname{CanAccessSM}(x).$

Solution2: Let the domain be people and mailboxes and use Access(*x*, *y*) as defined in the solution to part (a), and then also add InGroup(*x*) for "*x* is in the group", and let SystemMailBox be the name for the system mailbox.

FileSystemLocked $\rightarrow \forall x (InGroup(x) \rightarrow Access(x, SystemMailBox)).$

(c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.

Let the domain be all applications. Let Firewall(x) be "x is the firewall", and let ProxyServer(x) be "x is the proxy server." Let Diagnostic(x) be "x is in a diagnostic state".

$$\forall x \forall y ((Firewall(x) \land Diagnostic(x)) \rightarrow) ProxyServer(y) \rightarrow Diagnostic(y))$$

(d) At least one router is functioning normally if the throughput is between 100kbps and 500 kbps and the proxy server is not in diagnostic mode.

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 $\forall x \forall y ((Firewall(x) \land Diagnostic(x)) \rightarrow (ProxyServer(y) \rightarrow Diagnostic(y)))$

(d) At least one router is functioning normally if the throughput is between 100kbps and 500 kbps and the proxy server is not in diagnostic mode.

Let the domain be all applications and routers. Let Router(*x*) be "*x* is a router", and let ProxyServer(*x*) be "*x* is the proxy server." Let Diagnostic(*x*) be "*x* is in a diagnostic state". Let *p* be "the throughput is between 100kbps and 500 kbps". Let Functioning(*y*) be "*y* is functioning normally".

 $p \land \forall x (\neg ProxyServer(x) \lor \neg Diagnostic(x))) \rightarrow \exists y (Router(y) \land Functioning(y))$

~ (PS(x) / D(x))

Practice Final: 1. Regularly Irregular

Let $\Sigma = \{0, 1\}$. Prove that the language $L = \{x \in \Sigma^* : \#_0(x) < \#_1(x)\}$ is irregular.

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Suppose, for the sake of contradiction, that $L = \{x \in \Sigma^* : \#_0(x) < \#_1(x)\}$ is regular. Then there is a DFA *M* such that *M* accepts exactly *L*.

Let *S* = [TODO]

Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. [TODO].

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Consider the string z = [TODO].
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Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that xz = [TODO], so $xz \in L$ but yz = [TODO], so $yz \notin L$. Since q is can be only one of an accept or reject state, M does not actually recognize L. That's a contradiction!

Let $\Sigma = \{0, 1\}$. Prove that the language $L = \{x \in \Sigma^* : \#_0(x) < \#_1(x)\}$ is irregular.

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Consider the string $z = 1^{b}$.

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Define

$$T(n) = \begin{cases} n & \text{if } n = 0, 1\\ 4T\left(\lfloor \frac{n}{2} \rfloor\right) + n & \text{otherwise} \end{cases}$$

Prove that $T(n) \le n^3$ for $n \ge 3$

We go by strong induction on n. Let P(n) be "T(n) $\leq n^3$ " for $n \geq 3$. Base Cases.

Induction Hypothesis. Induction Step.

We go by strong induction on n. Let P(n) be "T(n) $\leq n^3$ " for $n \geq 3$. <u>Base Cases.</u> When n = 3: When n = 4: When n = 5: <u>Induction Hypothesis.</u> <u>Induction Step.</u>

We go by strong induction on n. Let P(n) be "T(n) \leq n³" for $n \geq 3$. <u>Base Cases.</u> When n = 3: $T(3) = 4T(\lfloor \frac{3}{2} \rfloor) + 3 = 4T(1) + 3 = 7 \leq 27 = 3^3$. When n = 4: $T(4) = 4T(\lfloor \frac{4}{2} \rfloor) + 4 = 4T(2) + 4 = 28 \leq 64 = 4^3$. When n = 5: $T(5) = 4T(\lfloor \frac{5}{2} \rfloor) + 5 = 4T(2) + 5 = 29 \leq 4^4$. Induction Hypothesis. Induction Step.

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We go by strong induction on n. Let P(n) be "T(n) $\leq n^3$ " for $n \geq 3$. <u>Base Cases.</u> When n = 3: $T(3) = 4T(|\frac{3}{2}|) + 3 = 4T(1) + 3 = 7 \le 27 = 3^3$. When n = 4: $T(4) = 4T(\left|\frac{4}{2}\right|) + 4 = 4T(2) + 4 = 28 \le 64 = 4^3$. TG2) = K3 When n = 5: $T(5) = 4T(|\frac{5}{2}|) + 5 = 4T(2) + 5 = 29 \le 4^4$. Induction Hypothesis. Suppose $P(3) \land P(4) \land \cdots \land P(k)$ for some $k \ge 5$. Induction Step. $T(k+1) = 4T\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right) + k+1,$ because $k + 1 \ge 2$. $\leq 4\left(\left|\frac{k+1}{2}\right|\right)^3 + k + 1,$ by IH. $\leq 4\left(\frac{k+1}{2}\right)^3 + k + 1,$ by def of floor. $= 4\left(\frac{(k+1)^3}{2^3}\right) + k + 1,$ by algebra. $=\frac{(k+1)^3}{2}+k+1,$ by algebra. $=\frac{(k+1)((k+1)^2+2)}{2},$ by algebra. $\leq \frac{(k+1)((k+1)^2 + (k+1)^2)}{2},$ because $(k+1)^2 \ge 2$. $= (k+1)^3$. by algebra

We go by strong induction on n. Let P(n) be "T(n) $\leq n^3$ " for $n \geq 3$. <u>Base Cases.</u> When n = 3: $T(3) = 4T(|\frac{3}{2}|) + 3 = 4T(1) + 3 = 7 \le 27 = 3^3$. When n = 4: $T(4) = 4T(\left|\frac{4}{2}\right|) + 4 = 4T(2) + 4 = 28 \le 64 = 4^3$. When n = 5: $T(5) = 4T(|\frac{5}{2}|) + 5 = 4T(2) + 5 = 29 \le 4^4$. Induction Hypothesis. Suppose P(3) \land P(4) $\land \cdots \land$ P(k) for some k \ge 5. Induction Step. $T(k+1) = 4T\left(\left|\frac{k+1}{2}\right|\right) + k + 1$, because $k + 1 \ge 2$. $\leq 4\left(\left|\frac{k+1}{2}\right|\right)^3 + k + 1,$ by IH. $\leq 4\left(\frac{k+1}{2}\right)^3 + k + 1,$ by def of floor. $= 4\left(\frac{(k+1)^3}{2^3}\right) + k + 1,$ by algebra. $=\frac{(k+1)^3}{2}+k+1,$ by algebra. $=\frac{(k+1)((k+1)^2+2)}{2},$ by algebra. $\leq \frac{(k+1)((k+1)^2 + (k+1)^2)}{2},$ because $(k+1)^2 \ge 2$. $= (k+1)^3$. by algebra

<u>Conclusion</u>. Thus, since the base case and induction step hold, the P(n) is true for $n \ge 3$.

Practice Final: 3. All The Machines!

Let $\Sigma = \{0, 1, 2\}$. Consider L = {w $\in \Sigma^{+}$: Every 1 in the string has at least one 0 before and after it}.

(a) Give a regular expression that represents A.

(b) Give a DFA that recognizes A.

(c) Give a CFG that generates A.



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 $(0 \cup 2)^* (0(0 \cup 1 \cup 2)^*0)^* (0 \cup 2)^*$

(c) Give a CFG that generates A.

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 $(0 \cup 2)^* (0(0 \cup 1 \cup 2)^*)^* (0 \cup 2)^*$

(c) Give a CFG that generates A.

 $S \rightarrow 0S \mid 2S \mid T$ $T \rightarrow 0R0T \mid X$ $R \rightarrow 0 \mid 1 \mid 2$ $X \rightarrow 0X \mid 2X \mid \varepsilon$

Practice Final: 4. Structural CFGs

Consider the following CFG: $S \rightarrow \varepsilon | SS | S1 | S01$. Another way of writing the recursive definition of this set, Q, is as follows:

- ε∈Q Busis step
- If $S \in Q$, then $S1 \in Q$ and $S01 \in Q$
- If $S, T \in Q$, then $ST \in Q$.

Prove, by structural induction that if $w \in Q$, then w has at least as many 1's as 0's

Practice Final: 4. Structural CFGs Solution Busis step: EEQ / recursive step: SIEQ, SOIEQ ASEQ STEQ VS, JEQ #o(w), #, [w), P(w)="#o(w) <#, (w)" Buse case: #o (2)=0, #, [2)=0, #o(E) <= #(E), P(E) 11+: Assume P(u), P(v) for arbitrary u, v EQ IS: Lot u, v be arbitrary element in Q case1: By 1H, PLu) holds, #o(a) 兰井, (u) $\#_{\theta}(u_1) = \#_{\theta}(u), \#_{\eta}(u_1) = \#_{\eta}(u) + 1$ $\#_{s}(u_{1}) = \#_{s}(u_{1}) \leq \#_{s}(u_{1}) \leq \#_{s}(u_{1}) + | = \#_{s}(u_{1})$ Ply,)

Practice Final: 4. Structural CFGs Solution Case 1: #o(Uo1) = #du) + 1, #, (Uo1) = #,(U) + 1 By1H: 井olu) (井, lu) 井。(uo1) ニ 井っ(い)+1 ニ 井(い)+1 ニ 井, (い01) Pluš PLW), PUV). P(UJI) $(asc]; #_{o}(uv) = #_{o}(u) + #_{o}(v), #, (uv) = #_{i}(u) + #_{i}(v)$ $B_{3}(H; #_{o}(u)) \leq #, (u), #_{o}(v) \leq #_{i}(v)$ $\#_{o}(nv) = \#_{o}(n) + \#_{o}(v) \leq \#_{o}(n) + \#_{o}(v)$ By priciple of structural induction, p(w) VWER Pluv)

Practice Final: BONUS Set Proof

A = {x : x \equiv k (mod 4)}, B = {x : x = 4r+k for some integer r}. Prove A = B for all integer k Let k be an arbitrary integer Let a be an arbitrary in A ACB $a \ge k \pmod{4} \rightarrow k \equiv a \pmod{4} \rightarrow 4 | a - k$ 4r=a-k for some integer r $4r+k=\alpha \rightarrow \alpha=4r+k$ 411-0 42 = K-9 acB $4(-r) = -(k-\alpha)$ 4(-r) = n - k

Practice Final: BONUS Set Proof

bEA ASB, BSA, ASB

Practice Final: 5. Tralse!

For each of the following answer True or False and give a short explanation of your answer.

- (a) Any subset of a regular language is also regular.
- (b) The set of programs that loop forever on at least one input is decidable.
- (c) If $\mathbb{R} \subseteq A$ for some set *A*, then *A* is uncountable.
- (d) If the domain of discourse is people, the logical statement
 ∃x (P(x) → ∀y (x ≠ y → ¬P(y))
 can be correctly translated as "There exists a unique person who has property P".
- (e) $\exists x (\forall y P(x, y)) \rightarrow \forall y (\exists x P(x, y))$ is true regardless of what predicate *P* is.

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(a) Any subset of a regular language is also regular.

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(b) The set of programs that loop forever on at least one input is decidable.

•

False

(c) If $\mathbb{R} \subseteq A$ for some set A, then A is uncountable.

True surjection $f: IN \rightarrow A$ Assume A is complable i f(i) A - A/Ri 0 2

(d) If the domain of discourse is people, the logical statement $\exists x (P(x) \rightarrow \forall y (x \neq y \rightarrow \neg P(y))$ can be correctly translated as "There exists a unique person who has property P".

(e) $\exists x (\forall y P(x, y)) \rightarrow \forall y (\exists x P(x, y)) \text{ is true regardless of what predicate } P \text{ is.}$

Practice Final: 6. Regularly Irregular

The following is the graph of a binary relation *R*.



(a) Draw the transitive-reflexive closure of *R*.

(b) Let $S = \{(X, Y) : X, Y \in \mathcal{P}(\mathbb{N}) \land X \subseteq Y\}$. Recall that R is antisymmetric iff $((a, b) \in R \land a \neq b) \rightarrow (b, a) \notin R$. Prove that S is antisymmetric.

(a) Draw the transitive-reflexive closure of *R*.



(b) Let $S = \{(X, Y) : X, Y \in \mathcal{P}(\mathbb{N}) \land X \subseteq Y\}$. Recall that R is antisymmetric iff $((a, b) \in R \land a \neq b) \rightarrow (b, a)$ Prove that S is antisymmetric.

> Suppose (a1b) 65 and a # b. by def of S, a c b. By def of C, there is some x e b where x & a. Then, b & a, so (b, a) & R. Therefore, S is on Hsymmetric.

Practice Final: 8. Modern DFAs



Practice Final: 8. Modern DFAs Solution

Let $\Sigma = \{0, 1, 2\}$. Construct a DFA that recognizes exactly strings with a 0 in all positions *i* where i%3 = 0.

That's All, Folks!

Any questions?