



# Relations And Graphs

CSE 311 Winter 2022  
Lecture 22

# Relations

## Relations

A (binary) relation from  $A$  to  $B$  is a subset of  $A \times B$

A (binary) relation on  $A$  is a subset of  $A \times A$

Wait what?

$\leq$  is a relation on  $\mathbb{Z}$ .

" $3 \leq 4$ " is a way of saying "3 relates to 4" (for the  $\leq$  relation)

$(3,4)$  is an element of the set that defines the relation.

# Relations, Examples

It turns out, they've been here the whole time

$<$  on  $\mathbb{R}$  is a relation

i.e.  $\{(x, y) : x < y \text{ and } x, y \in \mathbb{R}\}$ .

$=$  on  $\Sigma^*$  is a relation

i.e.  $\{(x, y) : x = y \text{ and } x, y \in \Sigma^*\}$

For your favorite function  $f$ , you can define a relation from its domain to its co-domain

i.e.  $\{(x, y) : f(x) = y\}$

" $x$  when squared gives  $y$ " is a relation

i.e.  $\{(x, y) : x^2 = y, x, y \in \mathbb{R}\}$

# Relations, Examples

Fix a universal set  $\mathcal{U}$ .

$\subseteq$  is a relation. What's it on?

$\mathcal{P}(\mathcal{U})$

The set of all subsets of  $\mathcal{U}$

# More Relations

$$R_1 = \{(a, 1), (a, 2), (b, 1), (b, 3), (c, 3)\}$$

Is a relation (you can define one just by listing what relates to what)

Equivalence mod 5 is a relation.

$$\{(x, y) : x \equiv y \pmod{5}\}$$

We'll also say "x relates to y if and only if they're congruent mod 5"

# Properties of relations

What do we do with relations? Usually we prove properties about them.

## Symmetry

A binary relation  $R$  on a set  $S$  is “symmetric” iff  
for all  $a, b \in S$ ,  $[(a, b) \in R \rightarrow (b, a) \in R]$

$=$  on  $\Sigma^*$  is symmetric, for all  $a, b \in \Sigma^*$  if  $a = b$  then  $b = a$ .

$\subseteq$  is not symmetric on  $\mathcal{P}(\mathcal{U})$  –  $\{1,2,3\} \subseteq \{1,2,3,4\}$  but  $\{1,2,3,4\} \not\subseteq \{1,2,3\}$

## Transitivity

A binary relation  $R$  on a set  $S$  is “transitive” iff  
for all  $a, b, c \in S$ ,  $[(a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R]$

$=$  on  $\Sigma^*$  is transitive, for all  $a, b, c \in \Sigma^*$  if  $a = b$  and  $b = c$  then  $a = c$ .

$\subseteq$  is transitive on  $\mathcal{P}(\mathcal{U})$  – for any sets  $A, B, C$  if  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ .

$\in$  is not a transitive relation –  $1 \in \{1,2,3\}$ ,  $\{1,2,3\} \in \mathcal{P}(\{1,2,3\})$  but  $1 \notin \mathcal{P}(\{1,2,3\})$ .

# Warm up

Show that  $a \equiv b \pmod{n}$  if and only if  $b \equiv a \pmod{n}$

$$a \equiv b \pmod{n} \leftrightarrow n \mid (b - a) \leftrightarrow nk = b - a \text{ (for } k \in \mathbb{Z}) \leftrightarrow$$

$$n(-k) = a - b \text{ (for } -k \in \mathbb{Z}) \leftrightarrow n \mid (a - b) \leftrightarrow b \equiv a \pmod{n}$$

This was a proof that the relation  $\{(a, b) : a \equiv b \pmod{n}\}$  is symmetric!

It was actually overkill to show if and only if. Showing just one direction turns out to be enough!

$a - b = (q - 1)n + (a \% n)$ . Observe that  $q - 1$  is an integer, and that this is the form of the division theorem for  $(a - b) \% n$ . Since the division theorem guarantees a unique integer,  $(a - b) \% n = (a \% n)$

# What about transitivity?

Some quarters there's a homework problem...we didn't have one this time.

Divides is a transitive relation!

If  $p|q$  and  $q|r$  then  $p|r$ .



# More Properties of relations

What do we do with relations? Usually we prove properties about them.

## Antisymmetry

A binary relation  $R$  on a set  $S$  is "antisymmetric" iff  
for all  $a, b \in S$ ,  $[(a, b) \in R \wedge a \neq b \rightarrow (b, a) \notin R]$

$\leq$  is antisymmetric on  $\mathbb{Z}$

## Reflexivity

A binary relation  $R$  on a set  $S$  is "reflexive" iff  
for all  $a \in S$ ,  $[(a, a) \in R]$

$\leq$  is reflexive on  $\mathbb{Z}$

$\leq$

# You've proven antisymmetry too!

(a) Prove that if  $a \mid b$  and  $b \mid a$ , where  $a$  and  $b$  are integers, then  $a = b$  or  $a = -b$ .

**Solution:**

Suppose that  $a \mid b$  and  $b \mid a$ , where  $a, b$  are integers. By the definition of divides, we have  $a \neq 0, b \neq 0$  and  $b = ka, a = jb$  for some integers  $k, j$ . Combining these equations, we see that  $a = j(ka)$ .

Then, dividing both sides by  $a$ , we get  $1 = jk$ . So,  $\frac{1}{j} = k$ . Note that  $j$  and  $k$  are integers, which is only possible if  $j, k \in \{1, -1\}$ . It follows that  $b = -a$  or  $b = a$ .

## Antisymmetry

A binary relation  $R$  on a set  $S$  is “antisymmetric” iff  
for all  $a, b \in S$ ,  $[(a, b) \in R \wedge a \neq b \rightarrow (b, a) \notin R]$

You showed  $\mid$  is antisymmetric on  $\mathbb{Z}^+$

for all  $a, b \in S$ ,  $[(a, b) \in R \wedge (b, a) \in R \rightarrow a = b]$  is equivalent to the definition in the box above

The box version is easier to understand, the other version is usually easier to prove.

# Try a few of your own

[Pollev.com/uwcse311](https://pollev.com/uwcse311)

Decide whether each of these relations are Reflexive, symmetric, antisymmetric, and transitive.

$\subseteq$  on  $\mathcal{P}(\mathcal{U})$

$\geq$  on  $\mathbb{Z}$

$>$  on  $\mathbb{R}$

$|$  on  $\mathbb{Z}^+$

$|$  on  $\mathbb{Z}$

$\equiv (\text{mod } 3)$  on  $\mathbb{Z}$

Symmetry: for all  $a, b \in S$ ,  $[(a, b) \in R \rightarrow (b, a) \in R]$

Antisymmetry: for all  $a, b \in S$ ,  $[(a, b) \in R \wedge a \neq b \rightarrow (b, a) \notin R]$

Transitivity: for all  $a, b, c \in S$ ,  $[(a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R]$

Reflexivity: for all  $a \in S$ ,  $[(a, a) \in R]$

# Try a few of your own

Symmetry: for all  $a, b \in S$ ,  $[(a, b) \in R \rightarrow (b, a) \in R]$

Antisymmetry: for all  $a, b \in S$ ,  $[(a, b) \in R \wedge a \neq b \rightarrow (b, a) \notin R]$

Transitivity: for all  $a, b, c \in S$ ,  
 $[(a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R]$

Reflexivity: for all  $a \in S$ ,  $[(a, a) \in R]$

Decide whether each of these relations are Reflexive, symmetric, antisymmetric, and transitive.

$\subseteq$  on  $\mathcal{P}(\mathcal{U})$  reflexive, antisymmetric, transitive

$\geq$  on  $\mathbb{Z}$  reflexive, antisymmetric, transitive

$>$  on  $\mathbb{R}$  antisymmetric, transitive

$|$  on  $\mathbb{Z}^+$  reflexive, antisymmetric, transitive

$|$  on  $\mathbb{Z}$  reflexive, transitive

$\equiv (\text{mod } 3)$  on  $\mathbb{Z}$  reflexive, symmetric, transitive

# Two Prototype Relations

A lot of fundamental relations follow one of two prototypes:

## Equivalence Relation

A relation that is reflexive, symmetric, and transitive is called an “equivalence relation”

## Partial Order Relation

A relation that is reflexive, antisymmetric, and transitive is called a “partial order”

# Equivalence Relations

Equivalence relations “act kinda like equals”

$\equiv \pmod{n}$  is an equivalence relation.

$\equiv$  on compound propositions is an equivalence relation.

Fun fact: Equivalence relations “partition” their elements.

An equivalence relation  $R$  on  $S$  divides  $S$  into sets  $S_1, \dots, S_k$  such that.

$\forall s (s \in S_i \text{ for some } i)$

$\forall s, s' (s, s' \in S_i \text{ for some } i \text{ if and only if } (s, s') \in R)$

$S_i \cap S_j = \emptyset$  for all  $i \neq j$

# Partial Orders

Partial Orders “behave kinda like less than or equal to”

In the sense that they put things in order

But it’s only kinda like less than – it’s possible that some elements can’t be compared.

$|$  on  $\mathbb{Z}^+$  is a partial order

$\subseteq$  on  $\mathcal{P}(\mathcal{U})$  is a partial order

$x$  is a prerequisite of (or-equal-to)  $y$  is a partial order on CSE courses

# Why Bother?

If you prove facts about all equivalence relations or all partial orders, you instantly get facts in lots of different contexts.

If you learn to recognize partial orders or equivalence relations, you can get **a lot** of intuition for new concepts in a short amount of time.

Why now? We'll want relations over the next few weeks (and it's a convenient way to review proving implications, for all statements, and so on)



# Combining Relations

Given a relation  $R$  from  $A$  to  $B$

And a relation  $S$  from  $B$  to  $C$ ,

The relation  $S \circ R$  from  $A$  to  $C$  is

$$\{(a, c) : \exists b[(a, b) \in R \wedge (b, c) \in S]\}$$

Yes, I promise it's  $S \circ R$  not  $R \circ S$  – it makes more sense if you think about relations  $(x, f(x))$  and  $(x, g(x))$

But also don't spend a ton of energy worrying about the order, we almost always care about  $R \circ R$ , where order doesn't matter.

# Combining Relations

To combine relations, it's a lot easier if we can see what's happening.

We'll use a representation of a directed graph

# Directed Graphs

$$G = (V, E)$$

$V$  is a set of vertices (an underlying set of elements)

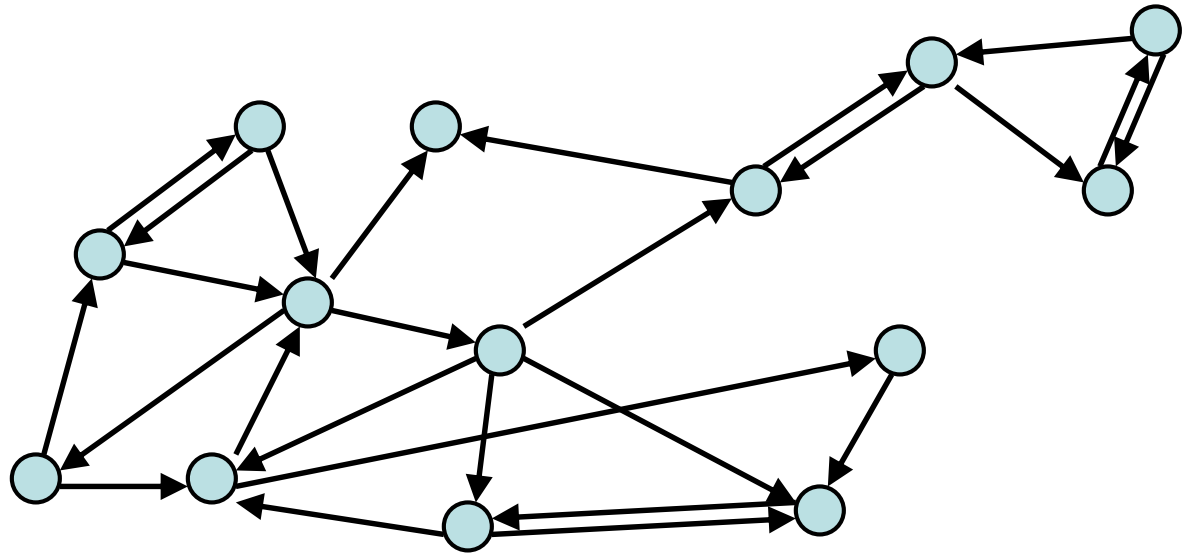
$E$  is a set of edges (ordered pairs of vertices; i.e. connections from one to the next).

**Path**  $v_0, v_1, \dots, v_k$  such that  $(v_i, v_{i+1}) \in E$

**Simple Path**: path with all  $v_i$  distinct

**Cycle**: path with  $v_0 = v_k$  (and  $k > 0$ )

**Simple Cycle**: simple path plus edge  $(v_k, v_0)$  with  $k > 0$



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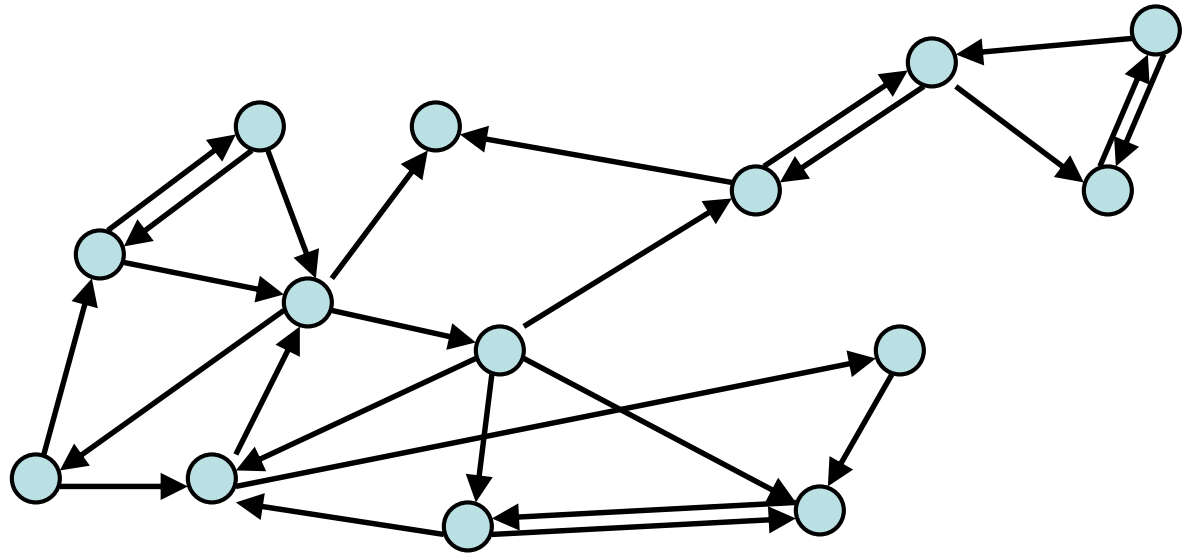
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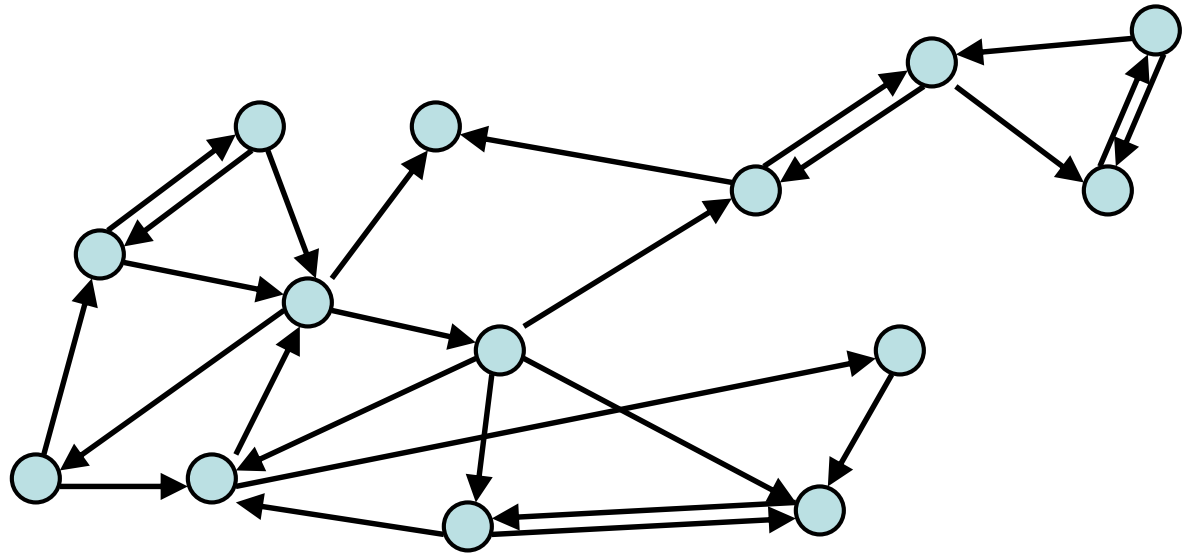
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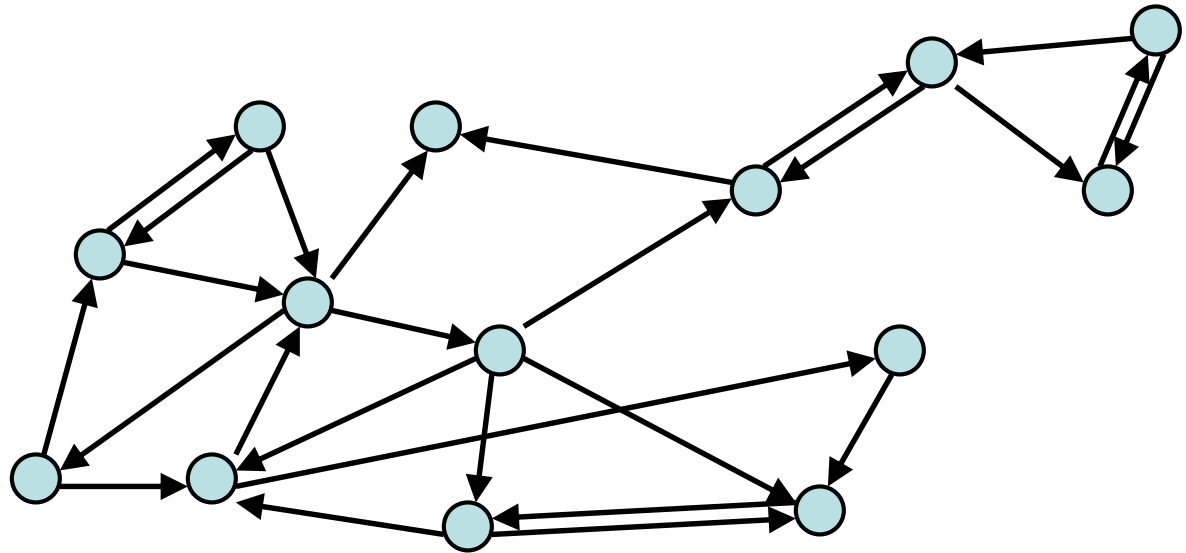
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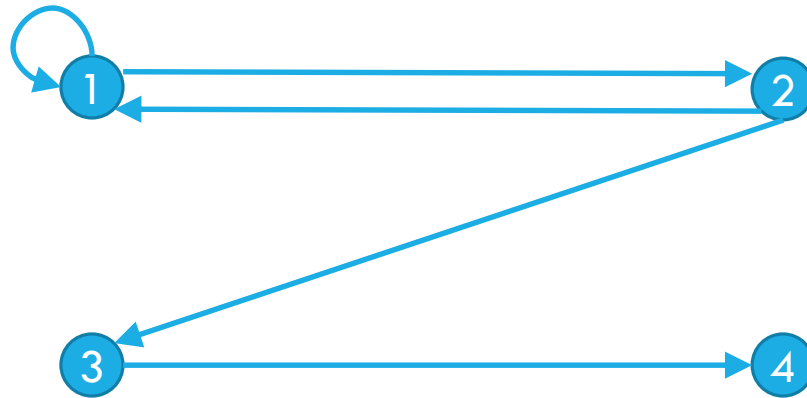
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# Representing Relations

To represent a relation  $R$  on a set  $A$ , have a vertex for each element of  $A$  and have an edge  $(a, b)$  for every pair in  $R$ .

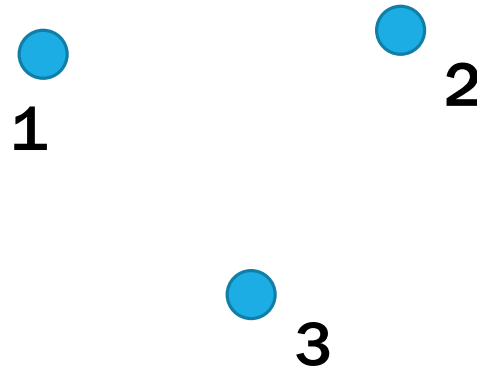
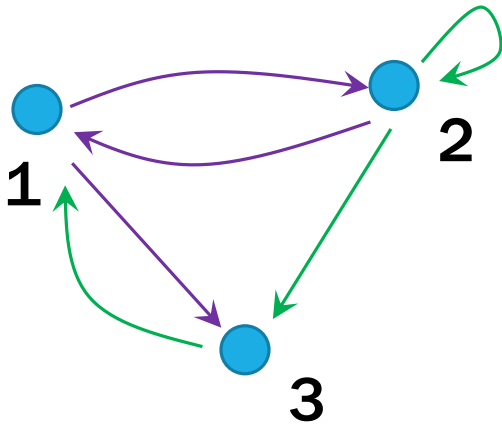
Let  $A$  be  $\{1, 2, 3, 4\}$  and  $R$  be  $\{(1, 1), (1, 2), (2, 1), (2, 3), (3, 4)\}$



# Combining Relations

If  $S = \{(2, 2), (2, 3), (3, 1)\}$  and  $R = \{(1, 2), (2, 1), (1, 3)\}$

Compute  $S \circ R$  i.e. every pair  $(a, c)$  with a  $b$  with  $(a, b) \in R$  and  $(b, c) \in S$

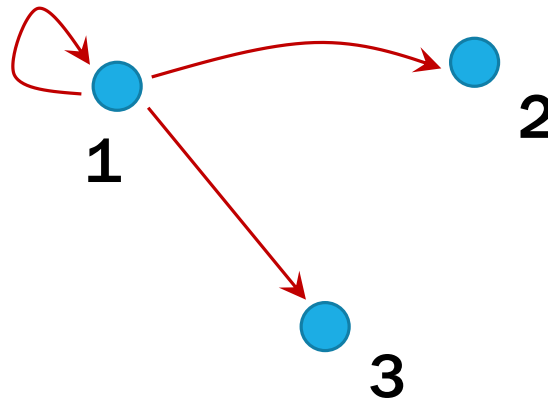
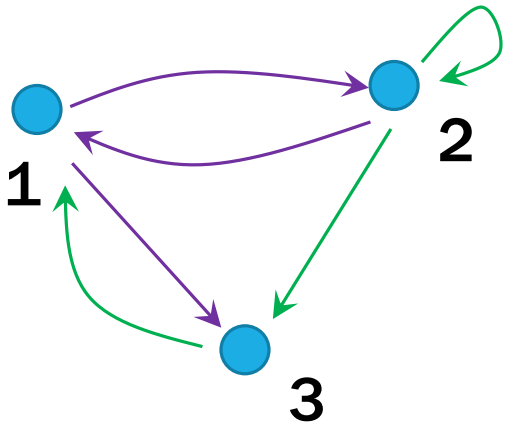




# Combining Relations

If  $S = \{(2, 2), (2, 3), (3, 1)\}$  and  $R = \{(1, 2), (2, 1), (1, 3)\}$

Compute  $S \circ R$  i.e. every pair  $(a, c)$  with a  $b$  with  $(a, b) \in R$  and  $(b, c) \in S$



# Combining Relations

Let  $R$  be a relation on  $A$ .

Define  $R^0$  as  $\{(a, a) : a \in A\}$

$$R^k = R^{k-1} \circ R$$

$(a, b) \in R^k$  if and only if there is a path of length  $k$  from  $a$  to  $b$  in  $R$ .

We can find that on the graph!

# More Powers of $R$ .

For two vertices in a graph,  $a$  can reach  $b$  if there is a path from  $a$  to  $b$ .

Let  $R$  be a relation on the set  $A$ . The connectivity relation  $R^*$  consists of all pairs  $(a, b)$  such that  $a$  can reach  $b$  (i.e. there is a path from  $a$  to  $b$  in  $R$ )

$$R^* = \bigcup_{k=0}^{\infty} R^k$$

Note we're starting from 0 (the textbook makes the unusual choice of starting from  $k = 1$ ).

# What's the point of $R^*$

$R^*$  is also the “reflexive-transitive closure of  $R$ .”

It answers the question “what’s the minimum amount of edges I would need to add to  $R$  to make it reflexive and transitive.”

Why care about that? The transitive-reflexive closure can be a summary of data – you might want to precompute it so you can easily check if  $a$  can reach  $b$  instead of recomputing it every time.

# Relations and Graphs

Describe how each property will show up in the graph of a relation.

Reflexive

Symmetric

Antisymmetric

Transitive

# Relations and Graphs

Describe how each property will show up in the graph of a relation.

Reflexive

Every vertex has a "self-loop" (an edge from the vertex to itself)

Symmetric

Every edge has its "reverse edge" (going the other way) also in the graph.

Antisymmetric

No edge has its "reverse edge" (going the other way) also in the graph.

Transitive

If there's a length-2 path from  $a$  to  $b$  then there's a direct edge from  $a$  to  $b$