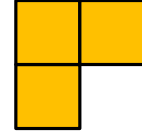


Today's problems on physical handout
(or on the webpage).

Even More Induction

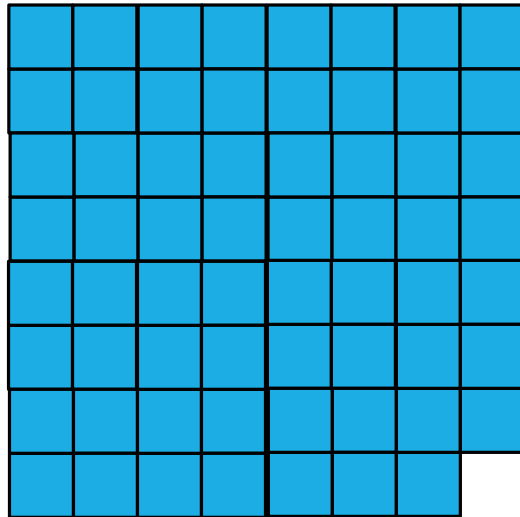
CSE 311 Winter 2022
Lecture 17

Gridding



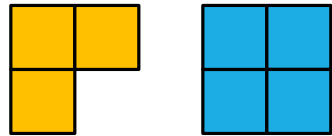
I've got a bunch of these 3 piece tiles.

I want to fill a $2^n \times 2^n$ grid ($n \geq 1$) with the pieces, except for a 1×1 spot in a corner.



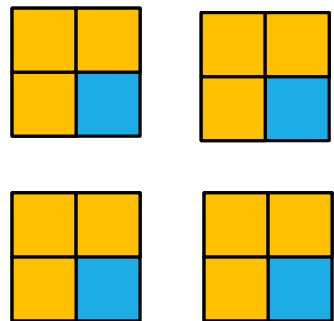
Gridding: Not a formal proof, just a sketch

Base Case: $n = 1$



Inductive hypothesis: Suppose you can tile a $2^k \times 2^k$ grid, except for a corner.

Inductive step: $2^{k+1} \times 2^{k+1}$, divide into quarters. By IH can tile...



Recursively Defined Functions

Just like induction works well with recursive code, it also works well for recursively-defined functions.

Define the Fibonacci numbers as follows:

$$f(0) = 1$$

$$f(1) = 1$$

$$f(n) = f(n - 1) + f(n - 2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

*This is a somewhat unusual definition, $f(0) = 0, f(1) = 1$ is more common.

Fibonacci Inequality

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Fibonacci Inequality

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

Define $P(n)$ to be " $f(n) \leq 2^n$ " We show $P(n)$ is true for all $n \geq 0$ by induction on n .

Base Cases: ($n = 0$): $f(0) = 1 \leq 1 = 2^0$.

($n = 1$): $f(1) = 1 \leq 2 = 2^1$.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \dots \wedge P(k)$ for an arbitrary $k \geq 1$.

Inductive step:

Target: $P(k+1)$. i.e. $f(k+1) \leq 2^{k+1}$

Fibonacci Inequality

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

Define $P(n)$ to be " $f(n) \leq 2^n$ ". We show $P(n)$ is true for all $n \geq 0$ by induction on n .

Base Cases: ($n = 0$): $f(0) = 1 \leq 1 = 2^0$.

($n = 1$): $f(1) = 1 \leq 2 = 2^1$.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \dots \wedge P(k)$ for an arbitrary $k \geq 1$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have $f(k+1) \leq 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$.

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

Claim: $3 \mid (2^{2n} - 1)$ for all $n \in \mathbb{N}$.

[Define $P(n)$]

Base Case

Inductive Hypothesis

Inductive Step

[conclusion]

Claim: $3 \mid (2^{2^n} - 1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be " $3 \mid (2^{2^n} - 1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2^n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3 \mid (2^{2 \cdot 0} - 1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step:

Target: $P(k + 1)$, i.e. $3 \mid (2^{2^{(k+1)}} - 1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3 \mid (2^{2n} - 1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be " $3 \mid (2^{2n} - 1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3 \mid (2^{2 \cdot 0} - 1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3 \mid (2^{2k} - 1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1$$

FORCE the expression in your IH to appear

Target: $P(k + 1)$, i.e. $3 \mid (2^{2(k+1)} - 1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3 \mid (2^{2n} - 1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be " $3 \mid (2^{2n} - 1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3 \mid (2^{2 \cdot 0} - 1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3 \mid (2^{2k} - 1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 4(2^{2k} - 1) + 4 - 1$$

By IH, we can replace $2^{2k} - 1$ with $3j$ for an integer j

$$2^{2(k+1)} - 1 = 4(3j) + 4 - 1 = 3(4j) + 3 = 3(4j + 1)$$

Since $4j + 1$ is an integer, we meet the definition of divides and we have:

Target: $P(k + 1)$, i.e. $3 \mid (2^{2(k+1)} - 1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3 \mid (2^{2^n} - 1)$ for all $n \in \mathbb{N}$.

That inductive step might still seem like magic.

It sometimes helps to run through examples, and look for patterns:

$$2^{2 \cdot 0} - 1 = 0 = 3 \cdot 0$$

$$2^{2 \cdot 1} - 1 = 3 = 3 \cdot 1$$

$$2^{2 \cdot 2} - 1 = 15 = 3 \cdot 5$$

$$2^{2 \cdot 3} - 1 = 63 = 3 \cdot 21$$

$$2^{2 \cdot 4} - 1 = 255 = 3 \cdot 85$$

$$2^{2 \cdot 5} - 1 = 1023 = 3 \cdot 341$$

The divisor goes from k to $4k + 1$

$$0 \rightarrow 4 \cdot 0 + 1 = 1$$

$$1 \rightarrow 4 \cdot 1 + 1 = 5$$

$$5 \rightarrow 4 \cdot 5 + 1 = 21$$

...

That might give us a hint that $4k + 1$ will be in the algebra somewhere, and give us another intermediate target.

Induction: Hats!

You have n people in a line ($n \geq 2$). Each of them wears either a **purple hat** or a **gold hat**. The person at the front of the line wears a purple hat. The person at the back of the line wears a gold hat.

Show that for every arrangement of the line satisfying the rule above, there is a person with a purple hat next to someone with a gold hat.

Yes this is kinda obvious. I promise this is good induction practice.

Yes you could argue this by contradiction. I promise this is good induction practice.

Induction: Hats!

Define $P(n)$ to be "in any line of gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n .

Base Case: $n = 2$

Inductive Hypothesis:

Inductive Step:

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be “in a line of gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat”

We show $P(n)$ for all integers $n \geq 2$ by induction on n .

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Target: there is someone in a purple hat next to someone in a gold hat.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be "in a line of gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n .

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Case 1: There is someone with a purple hat next to the person in the gold hat at one end. Then those people are the required adjacent opposite hats.

Case 2: There is a person with a gold hat next to the person in the gold hat at the end. Then the line from the second person to the end is length k , has a gold hat at one end and a purple hat at the other. Applying the inductive hypothesis, there is an adjacent, opposite-hat wearing pair.

In either case we have $P(k + 1)$.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Fibonacci Inequality Two

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

[Define $P(n)$]

Base Cases:

$$P(2): f(2) = f(1) + f(0) = 1 + 1 = 2$$

$$2^{2/2} = 2^1 = 2$$

$$f(2) = 2 \geq 2 = 2^{2/2}$$

Inductive Hypothesis:

Inductive step:

$$P(3): f(3) = f(2) + f(1) = 2 + 1 = 3$$

$$2^{3/2} = 2\sqrt{2}$$

$$3 \geq 2\sqrt{2}$$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

$$n \geq 2$$

Fibonacci Inequality Two

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be " $f(n) \geq 2^{n/2}$ " We show $P(n)$ is true for all $n \geq 2$ by induction on n .

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \dots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

Target: $f(k+1) \geq 2^{(k+1)/2}$

Therefore, we have $P(n)$ for all $n \geq 2$ by the principle of induction.

$n \geq 2$

Fibonacci Inequality Two

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be " $f(n) \geq 2^{n/2}$ " We show $P(n)$ is true for all $n \geq 2$ by induction on n .

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \dots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

$$f(k+1) \geq 2^{k/2} + 2^{(k-1)/2} = 2^{(k-1)/2} [2^{1/2} + 1] = 2^{(k-1)/2} \cdot 2 = 2^{(k+1)/2}$$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

Fibonacci Inequality Two

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be " $f(n) \geq 2^{n/2}$ " We show $P(n)$ is true for all $n \geq 2$ by induction on n .

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \dots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

$$f(k+1) \geq 2^{k/2} + 2^{(k-1)/2}$$

$$= 2^{(k-1)/2}(\sqrt{2} + 1)$$

$$\geq 2^{(k-1)/2} \cdot 2$$

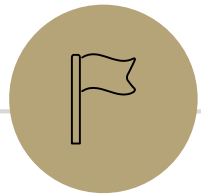
$$= 2^{(k+1)/2}$$

By IH, $f(k) \geq 2^{k/2}$

(factoring out)
 $(\sqrt{2} + 1 \approx 2.4 > 2)$

$P(k \wedge k-1)$:
 $f(k+1) \geq 2$

Therefore, we have $P(n)$ for all $n \geq 2$ by the principle of induction.



More Practice

Even More Induction Practice

$$\text{Let } g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases}$$

$$\text{Let } h(n) = n^n$$

Claim: $h(n) \geq g(n)$ for all integers $n \geq 1$

Even More Induction Practice

Define $P(n)$ to be " $h(n) \geq g(n)$ for all integers $n \geq 1$ "

We show $P(n)$ for all $n \geq 1$ by induction on n .

Base Case

Inductive Hypothesis:

Inductive Step:

Thus $P(k + 1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n .

$$\begin{aligned} \text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n \end{aligned}$$

Even More Induction Practice

Define $P(n)$ to be " $h(n) \geq g(n)$ for all integers $n \geq 1$ "

We show $P(n)$ for all $n \geq 1$ by induction on n .

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$g(k + 1) = (k + 1) \cdot g(k)$$

$$= (k + 1)^{k+1}.$$

Thus $P(k + 1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n .

$$\begin{aligned} \text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n \end{aligned}$$

Even More Induction Practice

Define $P(n)$ to be " $h(n) \geq g(n)$ for all integers $n \geq 1$ "

We show $P(n)$ for all $n \geq 1$ by induction on n .

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$\begin{aligned} g(k+1) &= (k+1) \cdot g(k) \\ &\leq (k+1) \cdot h(k) \text{ by IH.} \end{aligned}$$

$$= (k+1)^{k+1}.$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n .

$$\begin{aligned} \text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n \end{aligned}$$

Even More Induction Practice

Define $P(n)$ to be " $h(n) \geq g(n)$ for all integers $n \geq 1$ "

We show $P(n)$ for all $n \geq 1$ by induction on n .

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$\begin{aligned}g(k + 1) &= (k + 1) \cdot g(k) \\ &\leq (k + 1) \cdot h(k) && \text{by IH.} \\ &\leq (k + 1) \cdot k^k && \text{by definition of } h(k) \\ &\leq (k + 1) \cdot (k + 1)^k \\ &= (k + 1)^{k+1}.\end{aligned}$$

Thus $P(k + 1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n .

$$\begin{aligned}\text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n\end{aligned}$$

Even More Induction Practice

Define $P(n)$ to be " $h(n) \geq g(n)$ for all integers $n \geq 1$ "

We show $P(n)$ for all $n \geq 1$ by induction on n .

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$\begin{aligned} g(k+1) &= (k+1) \cdot g(k) \\ &\leq (k+1) \cdot h(k) && \text{by IH.} \\ &\leq (k+1) \cdot k^k && \text{by definition of } h(k) \\ &\leq (k+1) \cdot (k+1)^k \\ &= (k+1)^{k+1}. \end{aligned}$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n .

$$\begin{aligned} \text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n \end{aligned}$$

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^n 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n .

Base Case ($n = 0$): $\sum_{i=0}^0 2 + 3i = 2 + 3 \cdot 0 = 2 = \frac{1 \cdot 4}{2} = \frac{(0+1)(3 \cdot 0 + 4)}{2}$

Inductive Hypothesis:

Inductive Step:

[Conclusion]

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^n 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n .

Base Case ($n = 0$): $\sum_{i=0}^0 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3 \cdot 0 + 4)}{2}$

Inductive Hypothesis. Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

Target: $\sum_{i=0}^{k+1} 2 + 3i = \frac{([k+1]+1)(3[k+1]+4)}{2}$

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^n 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n .

Base Case ($n = 0$): $\sum_{i=0}^0 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3 \cdot 0 + 4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

$$\sum_{i=0}^{k+1} 2 + 3i = \left(\sum_{i=0}^k 2 + 3i \right) + (2 + 3(k+1)). \text{ By IH, we have:}$$

$$\sum_{i=0}^{k+1} 2 + 3i = \frac{(k+1)(3k+4)}{2} + 2 + 3k + 3 = ???$$

$$= \frac{([k+1] + 1)(3[k+1] + 4)}{2}$$

$$\sum_{i=0}^{k+1} 2 + 3i = \frac{(k+1+1)(3(k+1)+4)}{2}$$

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^n 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n .

Base Case ($n = 0$): $\sum_{i=0}^0 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3 \cdot 0 + 4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

$\sum_{i=0}^{k+1} 2 + 3i = (\sum_{i=0}^k 2 + 3i) + (2 + 3(k+1))$. By IH, we have:

$$\begin{aligned} \sum_{i=0}^{k+1} 2 + 3i &= \frac{(k+1)(3k+4)}{2} + 2 + 3k + 3 = \frac{3k^2 + 7k + 4}{2} + \frac{6k + 10}{2} = \frac{3k^2 + 13k + 14}{2} \\ &= \frac{(3k+7)(k+2)}{2} = \frac{([k+1]+1)(3[k+1]+4)}{2} \end{aligned}$$

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n .