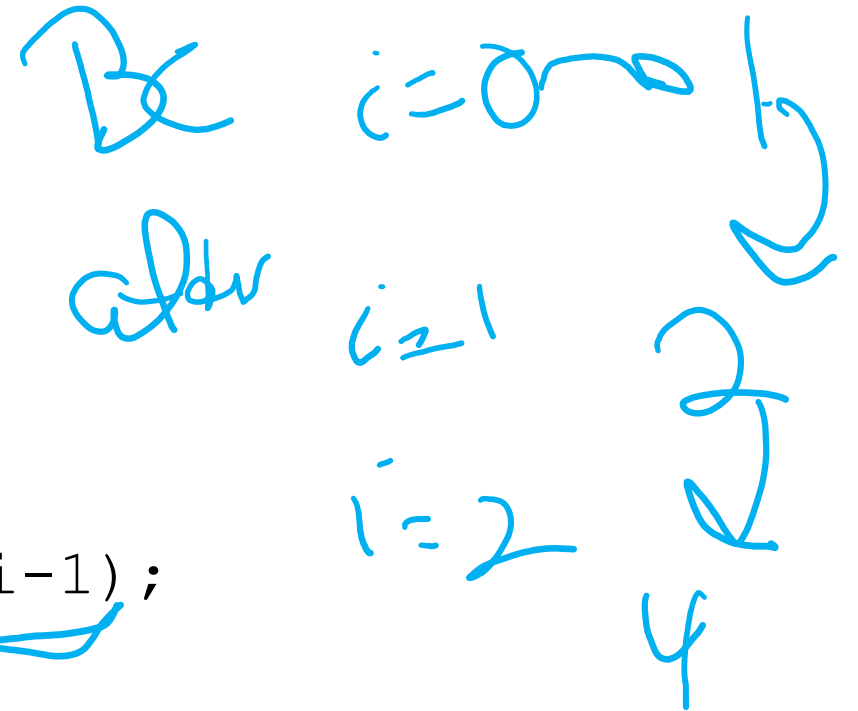


# Induction

CSE 311 Winter 2022  
Lecture 13

# How do we know recursion works?

```
//Assume i is a nonnegative integer
//returns 2^i.
public int CalculatesTwoToTheI(int i){
    if(i == 0)
        return 1;
    else
        return 2*CaclulatesTwoToTheI(i-1);
}
```



Why does `CalculatesTwoToTheI(4)` calculate  $2^4$ ?  
Convince the people around you!

$i=4$   $2^4 = 16$

# How do we know recursion works?

Something like this:

$$C(n) = 16$$

Well, as long as `CalculatesTwoToTheI(3) = 8`, we get 16...

Which happens as long as `CalculatesTwoToTheI(2) = 4`

Which happens as long as `CalculatesTwoToTheI(1) = 2`

Which happens as long as `CalculatesTwoToTheI(0) = 1`

And it is! Because that's what the base case says.

# How do we know recursion works?

There's really only two cases.

## The Base Case is Correct

CalculatesTwoToTheI(0) = 1 (which it should!)

And that means `CalculatesTwoToTheI(1) = 2`, (like it should)

And that means `CalculatesTwoToTheI(2) = 4`, (like it should)

And that means `CalculatesTwoToTheI(3) = 8`, (like it should)

And that means `CalculatesTwoToTheI(4) = 16`, (like it should)

IF the recursive call we make is correct  
THEN our value is correct.

# How do we know recursion works?

The code has two big cases,

So our proof had two big cases

“The base case of the code produces the correct output”

“IF the calls we rely on produce the correct output THEN the current call produces the right output”

# A bit more formally...

"The base case of the code produces the correct output"

"IF the calls we rely on produce the correct output THEN the current call produces the right output"

Let  $P(i)$  be "CalculatesTwoToTheI(i) returns  $2^i$ ."

How do we know  $P(4)$ ?

$P(0)$  is true.

And  $P(0) \rightarrow P(1)$ , so  $P(1)$ .

And  $P(1) \rightarrow P(2)$ , so  $P(2)$ .

And  $P(2) \rightarrow P(3)$ , so  $P(3)$ .

And  $P(3) \rightarrow P(4)$ , so  $P(4)$ .

# A bit more formally...

This works alright for  $P(4)$ .

What about  $P(1000)$ ?  $P(1000000000)$ ?

At this point, we'd need to show that implication  $P(k) \rightarrow P(k + 1)$  for A BUNCH of values of  $k$ .

But the code is the same each time.

And so was the argument!

We should instead show  $\forall k [P(k) \rightarrow P(k + 1)]$ .

# Induction

Your new favorite proof technique!

How do we show  $\forall n, P(n)$ ?

Show  $P(0)$

Show  $\forall k(P(k) \rightarrow P(k + 1))$

# Induction

$$\forall k \geq 0 (P(k) \rightarrow P(k+1))$$

```
//Assume i is a nonnegative integer
public int CalculatesTwoToTheI(int i){
    if(i == 0)
        return 1;
    else
        return 2*CaclulatesTwoToTheI(i-1);
}
```

Let  $P(i)$  be "CalculatesTwoToTheI( $i$ ) returns  $2^i$ ."

Note that if the input  $i$  is 0, then the if-statement evaluates to true, and  $1 = 2^0$  is returned, so  $P(0)$  is true.

Suppose  $P(k)$  holds for an arbitrary  $k \geq 0$ .

Consider the code run on  $k + 1$ . Since  $k \geq 0$ ,  $k + 1 > 0$  and we are in the else branch. By inductive hypothesis, CalculatesTwoToTheI( $k$ ) returns  $2^k$ , so the code run on  $k + 1$  returns  $2 \cdot 2^k = 2^{k+1}$ .

So  $P(k + 1)$  holds.

Therefore  $P(n)$  holds for all  $n \geq 0$  by the principle of induction.

# Making Induction Proofs Pretty

Let  $P(i)$  be the predicate "CalculatesTwoToTheI( $i$ ) returns  $2^i$ ." We prove  $P(n)$  holds for all  $n$  by induction on  $n$ .  $n, n < 0, n \geq 0$

**Base Case** ( $i = 0$ ) Note that if the input  $i$  is 0, then the if-statement evaluates to true, and  $1 = 2^0$  is returned, so  $P(0)$  is true.

**Inductive Hypothesis:** Suppose  $P(k)$  holds for an arbitrary  $k \geq 0$ .

**Inductive Step:** Since  $k \geq 0, k \geq 1$ , so the code goes to the recursive case. We will return  $2 \cdot \text{CalculatesTwoToTheI}(k)$ . By Inductive Hypothesis,

$\text{CalculatesTwoToTheI}(k) = 2^k$ . Thus we return  $2 \cdot 2^k = 2^{k+1}$ .

So  $P(k + 1)$  holds.

Therefore  $P(n)$  holds for all  $n \geq 0$  by the principle of induction.

# Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define  $P(n)$ . State that your proof is by induction on  $n$ .

2. Show  $P(0)$  i.e. show the base case

3. Suppose  $P(k)$  for an arbitrary  $k$ .

4. Show  $P(k + 1)$  (i.e. get  $P(k) \rightarrow P(k + 1)$ )

5. Conclude by saying  $P(n)$  is true for all  $n$  by induction.

# Some Other Notes

"by IH"

Always state where you use the inductive hypothesis when you're using it in the inductive step.

It's usually the key step, and the reader really needs to focus on it.

Be careful about what values you're assuming the Inductive Hypothesis for – the smallest possible value of  $k$  should assume the base case but nothing more.

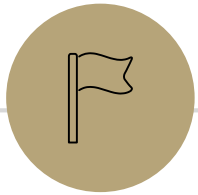
# The Principle of Induction (formally)

Domino

Principle of Induction

$$\begin{array}{c} P(0); \forall k (P(k) \rightarrow P(k + 1)) \\ \therefore \forall n (P(n)) \end{array}$$

Informally: if you knock over one domino, and every domino knocks over the next one, then all your dominoes fell over.



# Logical Ordering



# Logical Ordering

When doing a proof, we often work from both sides...

But we have to be careful!

When you read from top to bottom, every step has to follow only from what's **before** it, not after it.

Suppose our target is  $q$  and I know  $q \rightarrow p$  and  $r \rightarrow q$ .

What can I put as a "new target?"



# Logical Ordering

So why have all our prior steps been ok backward?

They've all been either:

A definition (which is always an "if and only if")

An algebra step that is an "if and only if"

Not all algebra steps are "if and only if"s!!

Even if your steps are "if and only if" you still have to put everything in order – start from your assumptions, and only assert something once it can be shown.

# A bad proof

Claim: if  $x$  is positive then  $x + 5 = -x - 5$ .

$$x + 5 = -x - 5$$

$$|x + 5| = |-x - 5|$$

$$|x + 5| = |-(x + 5)|$$

$$|x + 5| = |x + 5|$$

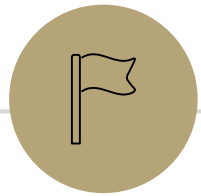
$$0 = 0$$

This claim is **false** – if you're trying to do algebra, you need to start with an equation you know (say  $x = x$  or  $2 = 2$  or  $0 = 0$ ) and expand to the equation you want.

$$5 = -5$$

$$|5| = |-5|$$

$F \rightarrow T$



**More induction!**

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# More Induction

Induction doesn't **only** work for code!

Show that  $\sum_{i=0}^n 2^i = 1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$ .

① define  $P(n)$  is " $\sum_{i=0}^n 2^i = 2^{n+1} - 1$ "

② BC: We want to show  $P(0)$

$$\sum_{i=0}^0 2^i = 2^0 = 1$$
$$2^{0+1} - 1 = 2 - 1 = 1$$

since LHS  
RHS are  
equal  $P(0)$  is true

# More Induction

Induction doesn't **only** work for code!

Show that  $\sum_{i=0}^n 2^i = 1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$ .

Let  $P(n) = \text{"}\sum_{i=0}^n 2^i = 2^{n+1} - 1\text{"}$

We show  $P(n)$  holds for all  $n$  by induction on  $n$ .

Base Case ( )

Inductive Hypothesis:

Inductive Step:

$P(n)$  holds for all  $n \geq 0$  by the principle of induction.

# More Induction

Induction doesn't **only** work for code!

Show that  $\sum_{i=0}^n 2^i = 1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$ .

Let  $P(n) = \text{"}\sum_{i=0}^n 2^i = 2^{n+1} - 1\text{"}$ .

We show  $P(n)$  holds for all  $n$  by induction on  $n$ .

Base Case ( $n = 0$ )  $\sum_{i=0}^0 2^i = 1 = 2 - 1 = 2^{0+1} - 1$ .

Inductive Hypothesis: Suppose  $P(k)$  holds for an arbitrary  $k \geq 0$ .

Inductive Step: We show  $P(k + 1)$ . Consider the summation  $\sum_{i=0}^{k+1} 2^i = 2^{k+1} + \sum_{i=0}^k 2^i = 2^{k+1} + 2^{k+1} - 1$ , where the last step is by IH.

Simplifying, we get:  $\sum_{i=0}^{k+1} 2^i = 2^{k+1} + 2^{k+1} - 1 = 2 \cdot 2^{k+1} - 1 = 2^{(k+1)+1} - 1$ .

$P(n)$  holds for all  $n \geq 0$  by the principle of induction.