Homework 6: Induction

Due date: Wednesday February 23rd at 10 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.

In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.

We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

In order to assist with the transition from formal proofs to English proofs, we've published a style guide on the website containing some tips. This guide contains references to proof materials that we haven't taught yet, so don't worry if some of these terms are unfamiliar.

Finally, be sure to read the grading guidelines for more information on what we're looking for.

Part I: Aim to complete this part by Wednesday Feb. 16

1. Well that just doesn't sound right [8 points]

Consider the following (very incorrect) induction proof:

① Let $P(n)$ be "$5n = 0$"
We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

② Base Case: $n = 0$
If $n = 0$ then $5n = 5 \cdot 0 = 0$, so $P(0)$ is true.

③ Inductive Hypothesis: Suppose $P(n)$ holds for $n = 0, \ldots, k$ for an arbitrary integer $k \geq 0$

④ Inductive Step:

Ⓐ We want to prove $P(k+1)$ is true, i.e. $5(k+1) = 0$.

Ⓑ Observe that $5(k+1) = 5(s) + 5(t)$. for integers s, t with $0 \leq s < k+1$ and $0 \leq t < k+1$.

Ⓒ Applying the inductive hypothesis twice, we have $5s = 0$ and $5t = 0$.

Ⓓ Substituting both into the original equation, we get: $5(k+1) = 0 + 0$, so $5(k+1) = 0$, as required.

⑤ The result follows for all $n \geq 0$ by induction.

(a) Find the smallest counterexample to the claim that $P(n)$ holds for all $n \in \mathbb{N}$. [3 points] You should both (1) show that your example is a counterexample and (2) argue why all smaller natural numbers are not counterexamples.

(b) Clearly identify the flaw in the proof; it will help to run through the proof with your smallest counterexample. For ease of explanation, we've taken the (unusual) step of labelling every sentence. [5 points]

2. Proof by contradiction [10 points]

Use proof by contradiction to show that for every prime number p, $\sqrt{p}$ is irrational. You may use the following fact (without needing to prove it): If p is a prime number and a and b are integers such that $p \mid (ab)$, then $p \mid a$ or $p \mid b$.

Part II: Some problems won't make sense when we release this homework

You **MUST** use induction for the proofs in Part II (unless the problem does not require a proof, or otherwise noted in the problem). You may use any appropriate version of induction (e.g. weak or strong or structural). Remember to define a predicate P as part of your proof.

3. Alligator Eats The Bigger One [20 points]

Prove that for all integers n with $n \geq 1$ we have $n \cdot 7^n \leq (n+13)!$.

4. Running Times [20 points]

You wrote a piece of recursive code. On an input of size n, your function takes $T(n)$ time to run, where:

$$T(n) = 5n \qquad \text{if } 1 \leq n \leq 4$$
$$T(n) = T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + 5n \qquad \text{for all } n > 4$$

In the definition above, $\lfloor x \rfloor$ is the "floor" function, it returns the greatest integer at most x.
For example: $\lfloor 3.2 \rfloor = 3$, $\lfloor 3.7 \rfloor = 3$, $\lfloor 3 \rfloor = 3$.

Show that for all $n \in \mathbb{N}$ with $n \geq 1$, $T(n) \leq 20n$

Hint 1: Notice that while $T()$ is defined with equality, you are only proving an inequality.
Hint 2: The only fact about the floor function you will need is $\lfloor x \rfloor$ is an integer and $1 \leq \lfloor x \rfloor \leq x$.

5. str000ng induction would be a good choice [20 points]

Let 0^n mean a string of n zeros. Let S be the set of strings defined as follows:

Basis Steps: $0^3 \in S$, $0^5 \in S$
Recursive Step: If $0^x, 0^y \in S$ then $0^x \cdot 0^y \in S$ where $\cdot$ is string concatenation.

Show that, for every integer $n \geq 12$ the set S contains the string 0^n.

Caution: Structural Induction is not the best tool for this problem. Structural induction shows $\forall x \in S(P(x))$. You're analyzing what the elements of S are in this problem, not proving a predicate holds for all elements of S.

6. Apples-to-Apples [23 points]

The Apple Picking Game is played between two players, who take turns removing apples from two bunches. Player 1 moves at the start of the game and Player 2 moves second. In each move, a player chooses one of the two bunches, then removes at least one apple from the bunch (as many as they choose, from a minimum of one to a maximum of all remaining apples in that bunch). Note that a player cannot take apple(s) from both bunches in a single turn.

The loser is the first player who is unable to remove any apples on their turn. That is, if there are no apples remaining at the start of the player's turn, they have lost the game.

Here is an example of how a game is played. Initially there are 2 apples in both bunches.
Move 1, player 1: removes 1 apple from bunch A.
Move 1, player 2: removes 1 apple from bunch A.
Move 2, player 1: removes 2 apples from bunch B.
Move 2, player 2: There are no apples remaining. Player 2 loses.

(a) Prove that player 2 can win any game of the Apple Picking Game, if both bunches contain the same number of apples at the start of the game. [20 points]
Hint: There is more than one reasonable choice for $P()$ here, think carefully about how your $P()$ relates to how you can do your inductive step, and how you get the overall claim.

(b) Describe the winning strategy for player 2. In other words, based on your proof, explain how player 2 should move in order to ensure they will win the game. You do not have to prove anything for this part. [3 points]

7. The Apple Doesn't Fall Far From The... `Tree` [20 points]

In CSE 143, you saw a recursive definition of trees. That definition looks a little different from what we saw in class.

The following definition is analogous to what you saw in 143. We'll call them `JavaTrees`.

Basis Step: `null` is a `JavaTree`.
Recursive Step: If L, R are `JavaTrees` then $(\texttt{data}, L, R)$ is also a `JavaTree`.

Show that for all `JavaTrees`: if they have k copies of `data` then they have $k + 1$ copies of `null`.

Remark: You're effectively showing here that a binary tree with k nodes has $k + 1$ `null` child pointers.

8. Find. The. Bug. [7 Points]

Recall the definition of **Trees** we used in class:
Basis Step: $\bullet$ is a **Tree**.
Recursive Step: If L and R are **Trees**, then $\texttt{Tree}(\bullet, L, R)$ is a **Tree**.

And recall the following definition of height:

height$(\bullet) = 0$
height $(\texttt{Tree}(\bullet, L, R)) = 1 + \max\{\text{height}(L), \text{height}(R)\}$

And the definition of leaves:
leaves$(\bullet) = 1$
leaves $(\texttt{Tree}(\bullet, L, R)) = \text{leaves}(L) + \text{leaves}(R)$

Your friend wants to show $\forall$**Trees** T, leaves$(T) = 2^{\text{height}(T)}$.

Your friend decided to use strong induction (structural induction would have been a better choice). Here is their proof.

① Define $P(n)$ to be: "all **Trees** of height n have 2^n leaves". We show $P(n)$ for all $n \geq 0$ by induction on n.

② Base Case ($n = 0$)
Ⓐ Consider an arbitrary tree of height 0, there is only one such tree $\bullet$.
Ⓑ leaves$(\bullet) = 1 = 2^0 = 2^{\text{height}(\bullet)}$.

③ Inductive Hypothesis: Suppose $P(0) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 0$.

④ Inductive Step:
Ⓐ Let T_1 and T_2 be arbitrary **Trees** of height k. By IH applied to each, we have: leaves$(T_1) = 2^k$, leaves$(T_2) = 2^k$.
Ⓑ Define $T = \texttt{Tree}(\bullet, T_1, T_2)$.
Ⓒ To make a **Tree** of height $k+1$, we must use the recursive rule.
Ⓓ Therefore, since T_1 and T_2 were arbitrary, T is an arbitrary **Tree** of its height.
Ⓔ height$(T) = 1 + \max\{k, k\} = k + 1$ and leaves$(T) = \text{leaves}(T_1) + \text{leaves}(T_2) = 2^k + 2^k = 2^{k+1}$
Ⓕ Since T is arbitrary **Tree** of height $k+1$, and it has $2^{\text{height}(T)} = 2^{k+1}$ leaves, we have $P(k+1)$

⑤ Therefore $P(n)$ is true for all $n \geq 0$ by the principle of induction.

⑥ Observe that every **Tree** has height at least 0, so we have for all **Trees** T, leaves$(T) = 2^{\text{height}(T)}$.

(a) The claim is false. Identify a counter-example. You should (1) draw (or otherwise describe) the example, (2) show that it is a **Tree**(e.g. by showing the rules to build it), (3) state its height and number of leaves. If you're using LaTeX, the command for the dot is `\bullet`. [3 points]

(b) Identify the biggest flaw in the proof. We have labeled the sentences to help you describe where it goes wrong. [4 points]