# Homework 2: Predicate Logic

Due date: Wednesday January 19th at 10 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

Be sure to read the grading guidelines on the assignments page for more information on what we're looking for.

## 1. Circuit du Soleil [10 points]

In this problem, we'll construct two propositions in terms of the variables $x, y, z$ and then use these propositions to build a circuit that computes a binary function $M(x, y, z)$.

(a) Give a propositional logic formula containing only the variables $x$ and $z$ which evaluates to $\neg x$ when $z$ is true and evaluates to false when $z$ is false. [2 points]

(b) Give a propositional logic formula containing only the variables $y$ and $z$ which evaluates to $y$ when $z$ is false and evaluates to false when $z$ is true. [2 points]

(c) Now consider the binary function $M(x, y, z)$ which is defined as:

$$M(x, y, 1) := \neg x$$

$$M(x, y, 0) := y$$

Draw a circuit that takes $x, y, z$ as input, uses only AND, OR, and NOT gates, and outputs $M(x, y, z)$. Your gates should not take more than two inputs. (Hint: combine your answers from (a) and (b)!) [6 points]

## 2. Think Contrapositive Be Contrapositive [14 points]

(a) If I go to the store and I cook for myself, then I will make soup.

(i) convert this sentence to propositional logic (as on homework 1, ensure you're giving variables to atomic propositions, not compound ones). [2 points]

(ii) take the contrapositive symbolically, and simplify so that $\neg$ signs are next to atomic propositions (i.e. only single variables). You are not required to show work for this part [2 points]

(iii) translate the contrapositive back to English. [3 points]

(b) In order to rent a car, it is necessary to have a driver's license.
Repeat steps (i)-(iii) from (a) for this sentence.

## 3. Some Symbols [10 points]

Prove that $(a \to b) \vee (c \to b) \equiv (a \wedge c) \to b$

For this problem, you need to write a symbolic proof using a chain of equivalences. To construct this proof, you should use propositional logic notation and rules (e.g. don't use the Boolean algebra reference sheet). You should also follow the symbolic proof guidelines.

Our proof has three "intermediate goals": convert to only ands/ors/nots with only atomic propositions negated, rearrange to eliminate the "extra" $b$, rearrange to final expression. Your proof is allowed to go differently (we will accept any correct, properly formatted proof), but our intermediate goals may help you if you are stuck.

## 4. Two of a kind [20 points]

(a) Translate the Boolean Algebra expression $((X \cdot (X + Y))' + X \cdot (Y + (X + Y')))'$ to Propositional Logic. Use the variables $a$ and $b$ to represent the propositions $X = 1$ and $Y = 1$, respectively. [2 points]

(b) Prove that your solution to (a) is a contradiction using a chain of equivalences. [16 points]

(c) Why do we know that the Boolean Algebra expression from part (a) is always 0? Explain. [2 points]

## 5. The New Normal Form [10 points]

Consider the following function $C(x, y, z)$ :

| $x$ | $y$ | $z$ | $C(x, y, z)$ |
|---|---|---|---|
| T | T | T | T |
| T | T | F | T |
| T | F | T | F |
| T | F | F | T |
| F | T | T | F |
| F | T | F | F |
| F | F | T | F |
| F | F | F | T |

(a) Express $C$ in Conjuntive Normal Form using Boolean Algebra notation. [5 points]

(b) Express $C$ in Disjunctive Normal Form using propositional logic notation. [5 points]

## 6. A tale of two $\exists$ [12 points]

Consider the following two expressions:

$$\exists x(\mathsf{P}(x) \land \mathsf{Q}(x)) \qquad \exists x\, \mathsf{P}(x) \land \exists x\, \mathsf{Q}(x)$$

(a) Give a domain of discourse and definitions of P and Q such that these expressions are **not** equivalent. Explain why your examples work (1-2 sentences). [6 points]

(b) Give a domain of discourse and definitions of P and Q such that these expressions **are** equivalent. Explain why your examples work (1-2 sentences). [6 points]

(c) **Extra Credit:** There is a logical relationship between these two expressions (one that is true for all domains and all predicates P, Q). By "logical relationship" we mean there is a logical connective that can join the two expressions together into a single true expression. What is that combined expression? Very briefly summarize why the relationship is true (1-2 sentences).

## 7. $\forall$BCs of NFTs [12 points]

With cryptocurrency on the rise, Robbie wants to learn about NFTs. He learns a couple facts that prepare him to go into the cryptocurrency world.

The predicates `Robbie`, `NFT`, and `Art` are true if and only if the input is Robbie, an NFT, or Art, respectively — for example, the predicate $\texttt{Robbie}(x)$ means "$x$ is Robbie." We define $\texttt{OnBlockchain}(x)$ to mean "$x$ is on the blockchain". Finally, the predicate $\texttt{Loves}(x, y)$ means "$x$ loves $y$" (note that the order of the parameters matters!).

Robbie hands you the following observations; translate them into English.

You should try to make your translations sound natural when possible, but you should not otherwise simplify the expression before translating (e.g. you would not apply any rules from the logical equivalences sheet before translating).

We will discuss "restricting the domain" after this homework is released, you may use these principles to make your translations more natural, but are not required to on *this* homework.

(a) $\exists x\,(\texttt{NFT}(x) \wedge \texttt{Art}(x))$

(b) $\forall x\,(\texttt{NFT}(x) \rightarrow \texttt{OnBlockchain}(x))$

(c) $\forall x \forall y\,((\texttt{Robbie}(x) \wedge \texttt{NFT}(y)) \rightarrow \texttt{Loves}(x, y))$

## 8. Extra Credit

Computers have storage spaces called "registers" (they are placed right near the processing unit to hold the values urgently needed for upcoming calculations). A register is a fixed number of bits long (i.e. a fixed number of `T` or `F`). For any two bits $a, b$ we define $\texttt{XNOR}(a, b) := \neg(a \oplus b)$.

Suppose you have two memory registers $R_i$ and $R_j$. You have only one operation available: $\texttt{XNOR}(R_i, R_j)$ performs `XNOR` bit-by-bit and **stores the result back in** $R_i$. By "bit-by-bit" we mean we `XNOR` the $k^{\text{th}}$ bit of $R_i$ with the $k^{\text{th}}$ bit of $R_j$ to get the $k^{\text{th}}$ bit of the result).

Show that you can swap the contents of $R_i$ and $R_j$ using only `XNOR` operations and **only** the registers $R_i, R_j$ – you are not allowed any "temporary variables" or other registers. Give both a list of steps and a brief explanation of how your solution works.