1. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$. **Solution:**

Let $P(n)$ be “$6n + 6 < 2^n$”. We will prove $P(n)$ for all integers $n \geq 6$ by induction on $n$.

**Base Case** ($n = 6$): $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

**Inductive Hypothesis:** Assume that $6k + 6 < 2^k$ for an arbitrary integer $k \geq 6$.

**Inductive Step:** Goal: Show $6(k + 1) + 6 < 2^{k+1}$

\[
6(k + 1) + 6 = 6k + 6 + 6
\]

\[
< 2^k + 6 \quad \text{[Inductive Hypothesis]}
\]

\[
< 2^k + 2^k \quad \text{[Since $2^k > 6$, since $k \geq 6$]}
\]

\[
= 2 \cdot 2^k
\]

\[
= 2^{k+1}
\]

So $P(k) \rightarrow P(k + 1)$ for an arbitrary integer $k \geq 6$.

**Conclusion:** $P(n)$ holds for all integers $n \geq 6$ by the principle of induction.

2. Induction with Formulas

These problems are a little more difficult and abstract. Try making sure you can do all the other problems before trying these ones.

(a) (i) Show that given two sets $A$ and $B$ that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (Don’t use induction.)

**Solution:**

Let $x$ be arbitrary. Then,

\[
x \in \overline{A \cup B} \equiv \neg (x \in A \cup B) \quad \text{[Definition of complement]}
\]

\[
\equiv \neg (x \in A \lor x \in B) \quad \text{[Definition of union]}
\]

\[
\equiv \neg (x \in A) \land \neg (x \in B) \quad \text{[De Morgan’s Laws]}
\]

\[
\equiv x \in \overline{A} \land x \in \overline{B} \quad \text{[Definition of complement]}
\]

\[
\equiv x \in (\overline{A} \cap \overline{B}) \quad \text{[Definition of intersection]}
\]

Since $x$ was arbitrary we have that $x \in \overline{A \cup B}$ if and only if $x \in \overline{A} \cap \overline{B}$ for all $x$. By the definition of set equality we’ve shown, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

(ii) Show using induction that for an integer $n \geq 2$, given $n$ sets $A_1, A_2, \ldots, A_{n-1}, A_n$ that

\[
\overline{A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n}
\]

**Solution:**
Let \( P(n) \) be “given \( n \) sets \( A_1, A_2, \ldots, A_{n-1}, A_n \) it holds that \( \overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n} \).” We show \( P(n) \) for all integers \( n \geq 2 \) by induction on \( n \).

**Base Case:** \( P(2) \) says that for two sets \( A_1 \) and \( A_2 \) that \( \overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2} \), which is exactly part (a) so \( P(2) \) holds.

**Inductive Hypothesis:** Suppose that \( P(k) \) holds for some arbitrary integer \( k \geq 2 \).

**Inductive Step:** Let \( A_1, A_2, \ldots, A_k, A_{k+1} \) be sets. Then by part (a) we have,

\[
(\overline{A_1 \cup A_2 \cup \cdots \cup A_k}) \cup A_{k+1} = \overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap A_{k+1}.
\]

By the inductive hypothesis we have \( \overline{A_1 \cup A_2 \cup \cdots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \). Thus,

\[
A_1 \cup A_2 \cup \cdots \cup A_k \cap A_{k+1} = (\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}) \cap A_{k+1}.
\]

We’ve now shown

\[
A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}}.
\]

which is exactly \( P(k+1) \).

**Conclusion** \( P(n) \) holds for all integers \( n \geq 2 \) by the principle of induction.

---

(b) (i) Show that given any integers \( a, b, \) and \( c \), if \( c \mid a \) and \( c \mid b \), then \( c \mid (a+b) \). (Don’t use induction.)

**Solution:**

Let \( a, b, \) and \( c \) be arbitrary integers and suppose that \( c \mid a \) and \( c \mid b \). Then by definition there exist integers \( j \) and \( k \) such that \( a = jc \) and \( b = kc \). Then \( a + b = jc + kc = (j+k)c \). Since \( j + k \) is an integer, by definition we have \( c \mid (a+b) \).

(ii) Show using induction that for any integer \( n \geq 2 \), given \( n \) numbers \( a_1, a_2, \ldots, a_{n-1}, a_n \), for any integer \( c \) such that \( c \mid a_i \), for \( i = 1, 2, \ldots, n \), that

\[
c \mid (a_1 + a_2 + \cdots + a_{n-1} + a_n).
\]

In other words, if a number divides each term in a sum then that number divides the sum.

**Solution:**

Let \( P(n) \) be “given \( n \) numbers \( a_1, a_2, \ldots, a_{n-1}, a_n \), for any integer \( c \) such that \( c \mid a_i \) for \( i = 1, 2, \ldots, n \), it holds that \( c \mid (a_1 + a_2 + \cdots + a_n) \).” We show \( P(n) \) holds for all integer \( n \geq 2 \) by induction on \( n \).

**Base Case:** \( P(2) \) says that given two integers \( a_1 \) and \( a_2 \), for any integer \( c \) such that \( c \mid a_1 \) and \( c \mid a_2 \) it holds that \( c \mid (a_1 + a_2) \). This is exactly part (a) so \( P(2) \) holds.

**Inductive Hypothesis:** Suppose that \( P(k) \) holds for some arbitrary integer \( k \geq 2 \).

**Inductive Step:** Let \( a_1, a_2, \ldots, a_k, a_{k+1} \) be \( k+1 \) integers. Let \( c \) be arbitrary and suppose that \( c \mid a_i \) for \( i = 1, 2, \ldots, k+1 \). Then we can write

\[
a_1 + a_2 + \cdots + a_k + a_{k+1} = (a_1 + a_2 + \cdots + a_k) + a_{k+1}.
\]

The sum \( a_1 + a_2 + \cdots + a_k \) has \( k \) terms and \( c \) divides all of them, meaning we can apply the inductive hypothesis. It says that \( c \mid (a_1 + a_2 + \cdots + a_k) \). Since \( c \mid (a_1 + a_2 + \cdots + a_k) \) and \( c \mid a_{k+1} \), by part (a) we have,

\[
c \mid (a_1 + a_2 + \cdots + a_k + a_{k+1}).
\]
This shows $P(k+1)$.

**Conclusion:** $P(n)$ holds for all integers $n \geq 2$ by induction the principle of induction.

### 3. Structural Induction

(a) Consider the following recursive definition of strings.

**Basis Step:** "" is a string

**Recursive Step:** If $X$ is a string and $c$ is a character then $\text{append}(c, X)$ is a string.

Recall the following recursive definition of the function $\text{len}$:

$$
\begin{align*}
\text{len}("") & = 0 \\
\text{len}(\text{append}(c, X)) & = 1 + \text{len}(X)
\end{align*}
$$

Now, consider the following recursive definition:

$$
\begin{align*}
\text{double}("") & = "" \\
\text{double}(\text{append}(c, X)) & = \text{append}(c, \text{append}(c, \text{double}(X))).
\end{align*}
$$

Prove that for any string $X$, $\text{len}(\text{double}(X)) = 2\text{len}(X)$.

**Solution:**

For a string $X$, let $P(X)$ be "$\text{len}(\text{double}(X)) = 2\text{len}(X)$". We prove $P(X)$ for all strings $X$ by structural induction on $X$.

**Base Case ($X = ""$):** By definition, $\text{len}(\text{double}("")) = \text{len}(\text{double}("")) = 0 = 2 \cdot 0 = 2\text{len}(\text{double}(""))$, so $P(\text{double}(""))$ holds

**Inductive Hypothesis:** Suppose $P(X)$ holds for some arbitrary string $X$.

**Inductive Step:** [Goal: Show that $P(\text{append}(c, X))$ holds for any character $c$.

$$
\begin{align*}
\text{len}(\text{double}(\text{append}(c, X))) & = \text{len}(\text{append}(c, \text{append}(c, \text{double}(X)))) \\
& = 1 + \text{len}(\text{append}(c, \text{double}(X))) \\
& = 1 + 1 + \text{len}(\text{double}(X)) \\
& = 2 + \text{len}(X) \\
& = 2(1 + \text{len}(X)) \\
& = 2(\text{len}(\text{append}(c, X)))
\end{align*}
$$

[By Definition of double]  
[By Definition of len]  
[By Definition of len]  
[By IH]  
[Algebra]  
[By Definition of len]

This proves $P(\text{append}(c, X))$.

**Conclusion:** $P(X)$ holds for all strings $X$ by structural induction.

(b) Consider the following definition of a (binary) Tree:

**Basis Step:** • is a Tree.

**Recursive Step:** If $L$ is a Tree and $R$ is a Tree then $\text{Tree}(\bullet, L, R)$ is a Tree.

The function $\text{leaves}$ returns the number of leaves of a Tree. It is defined as follows:

$$
\begin{align*}
\text{leaves}(\bullet) & = 1 \\
\text{leaves}(\text{Tree}(\bullet, L, R)) & = \text{leaves}(L) + \text{leaves}(R)
\end{align*}
$$

Also, recall the definition of size on trees:

\[
\begin{align*}
\text{size}(\bullet) &= 1 \\
\text{size}(\text{Tree}(\bullet, L, R)) &= 1 + \text{size}(L) + \text{size}(R)
\end{align*}
\]

Prove that \(\text{leaves}(T) \geq \text{size}(T)/2 + 1/2\) for all Trees \(T\).

**Solution:**

For a tree \(T\), let \(P(T)\) be \(\text{leaves}(T) \geq \text{size}(T)/2 + 1/2\). We prove \(P(T)\) for all trees \(T\) by structural induction on \(T\).

**Base Case (\(T = \bullet\):** By definition of \(\text{leaves}(\bullet)\), \(\text{leaves}(\bullet) = 1\) and \(\text{size}(\bullet) = 1\). So, \(\text{leaves}(\bullet) = 1 \geq 1/2 + 1/2 = \text{size}(\bullet)/2 + 1/2\), so \(P(\bullet)\) holds.

**Inductive Hypothesis:** Suppose \(P(L)\) and \(P(R)\) hold for some arbitrary trees \(L, R\).

**Inductive Step:** \(\text{Goal: Show that } P(\text{Tree}(\bullet, L, R))\) holds.

\[
\text{leaves}(\text{Tree}(\bullet, L, R)) = \text{leaves}(L) + \text{leaves}(R)
\]

\[
\geq (\text{size}(L)/2 + 1/2) + (\text{size}(R)/2 + 1/2) \quad \text{[By IH]}
\]

\[
= (1/2 + \text{size}(L)/2 + \text{size}(R)/2) + 1/2 \quad \text{[By Algebra]}
\]

\[
= \frac{1 + \text{size}(L) + \text{size}(R)}{2} + 1/2 \quad \text{[By Algebra]}
\]

\[
= \text{size}(T)/2 + 1/2 \quad \text{[By Definition of size]}
\]

This proves \(P(\text{Tree}(\bullet, L, R))\).

**Conclusion:** Thus, \(P(T)\) holds for all trees \(T\) by structural induction.

(c) Prove the previous claim using strong induction. Define \(P(n)\) as “all trees \(T\) of size \(n\) satisfy \(\text{leaves}(T) \geq \text{size}(T)/2 + 1/2\).” You may use the following facts:

- For any tree \(T\) we have \(\text{size}(T) \geq 1\).
- For any tree \(T\), \(\text{size}(T) = 1\) if and only if \(T = \bullet\).

If we wanted to prove these claims, we could do so by structural induction.

Note, in the inductive step you should start by letting \(T\) be an arbitrary tree of size \(k + 1\).

**Solution:**

Let \(P(n)\) be “all trees \(T\) of size \(n\) satisfy \(\text{leaves}(T) \geq \text{size}(T)/2 + 1/2\).” We show \(P(n)\) for all integers \(n \geq 1\) by strong induction on \(n\).

**Base Case:** Let \(T\) be an arbitrary tree of size 1. The only tree with size 1 is \(\bullet\), so \(T = \bullet\). By definition, 

\(\text{leaves}(T) = \text{leaves}(\bullet) = 1\) and thus \(\text{size}(T) = 1 = 1/2 + 1/2 = \text{size}(T)/2 + 1/2\). This shows the base case holds.

**Inductive Hypothesis:** Suppose that \(P(j)\) holds for all integers \(j = 1, 2, \ldots, k\) for some arbitrary integer \(k \geq 1\).

**Inductive Step:** Let \(T\) be an arbitrary tree of size \(k + 1\). Since \(k + 1 > 1\), we must have \(T \neq \bullet\). It follows from the definition of a tree that \(T = \text{Tree}(\bullet, L, R)\) for some trees \(L\) and \(R\). By definition, we have 

\(\text{size}(T) = 1 + \text{size}(L) + \text{size}(R)\). Since sizes are non-negative, this equation shows \(\text{size}(T) > \text{size}(L)\) and \(\text{size}(T) > \text{size}(R)\) meaning we can apply the inductive hypothesis. This says that \(\text{leaves}(L) \geq \text{size}(L)/2 + 1/2\) and \(\text{leaves}(R) \geq \text{size}(R)/2 + 1/2\).
We have,

\[
\text{leaves}(T) = \text{leaves}(\text{Tree}(*, L, R))
\]
\[
= \text{leaves}(L) + \text{leaves}(R) \quad \text{[By Definition of leaves]}
\]
\[
\geq (\text{size}(L)/2 + 1/2) + (\text{size}(R)/2 + 1/2) \quad \text{[By IH]}
\]
\[
= (1/2 + \text{size}(L)/2 + \text{size}(R)/2) + 1/2 \quad \text{[By Algebra]}
\]
\[
= \frac{1 + \text{size}(L) + \text{size}(R)}{2} + 1/2 \quad \text{[By Algebra]}
\]
\[
= \text{size}(T)/2 + 1/2 \quad \text{[By Definition of size]}
\]

This shows \(P(k + 1)\).

**Conclusion:** \(P(n)\) holds for all integers \(n \geq 1\) by the principle of strong induction.

Note, this proves the claim for all trees because every tree \(T\) has some size \(s \geq 1\). Then \(P(s)\) says that all trees of size \(s\) satisfy the claim, including \(T\).