Section 6

CSE 311 - Sp 2022

Administrivia

Announcements and Reminders

- HW5 due yesterday (BOTH PARTS) 10PM on Gradescope
  - Final late due date is Saturday 5/7 @ 10pm
  - We will do our best to release grades for part 2 immediately after the late due date Saturday night!
- HW4 grades out now
  - Regrade requests are open for one week
  - If you think your work may have been graded incorrectly, please submit a regrade request!
- HW6 is will be released on Monday!
  - You have slightly longer than a regular homework, it's due Wednesday 5/18 @ 10pm
- Midterm is This Weekend! (Friday 5/6 - Sunday 5/8)
  - "Take home" exam on Gradescope
  - You will have 2 hours to complete it, starting from when you open it on Gradescope
  - It is designed to take ~30 minutes

References

- How to LaTeX
  - https://courses.cs.washington.edu/courses/cse311/22sp/assignments/HowToLaTeX.pdf
- Logical Equivalences
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/reference-logical_equiv.pdf
- Inference Rules
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/InferenceRules.pdf
- Set Definitions
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/reference-sets.pdf
- Modular Arithmetic Definitions and Properties
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/reference-number-theory.pdf
- Induction Templates
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/induction-templates.pdf

Strong Induction

Problem 3 - Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

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How many rabbits does he have each year? Let's do some calculations, and see if we can find a pattern. Then, we'll try to prove the pattern holds for all n!

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$f(0) = 0$
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How many rabbits does he have each year? Let's do some calculations, and see if we can find a pattern. Then, we'll try to prove the pattern holds for all n!

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It seems like we have a pattern here!

$f(n) = n$

But we don't want to check for EVERY n, so let's see if we can prove it instead!

Strong Induction Template (also on course website!)

Let **P(n)** be "(whatever you're trying to prove)"
We show **P(n)** holds for all $\mathbf{n \geq b_{min}}$ by strong induction on **n**.

<u>Base Cases:</u> Show $\mathbf{P(b_{min}),\ P(b_{min+1}), \ldots, P(b_{max})}$ are all true

<u>Inductive Hypothesis:</u> Suppose $\mathbf{P(b_{min}) \land P(b_{min+1}) \land \ldots \land P(k)}$ holds for an arbitrary $\mathbf{k \geq b_{max}}$

<u>Inductive Step:</u> Show **P(k + 1)** (i.e. get $\mathbf{[P(b_{min}) \land P(k)] \rightarrow P(k+1)}$)

<u>Conclusion:</u> Therefore, **P(n)** holds for all $\mathbf{n \geq b_{min}}$ by the principle of strong induction.

Problem 3 - Cantelli’s Rabbits

Let **P(n)** be “” for all **n.**
We show **P(n)** holds for all **n** by strong induction on **n**.

Base Cases:

Inductive Hypothesis:

Inductive Step:

Conclusion:

Problem 3 - Cantelli's Rabbits

Let **P(n)** be "f(n) = n" for all **n ≥ 0.**
We show **P(n)** holds for all **n ≥ 0** by strong induction on **n**.

<u>Base Cases:</u>

<u>Inductive Hypothesis:</u>

<u>Inductive Step:</u>

<u>Conclusion:</u>

Problem 3 - Cantelli's Rabbits

Let **P(n)** be "f(n) = n" for all **n ≥ 0.**
We show **P(n)** holds for all **n ≥ 0** by strong induction on **n**.

Base Cases: (n = 0, n = 1): f(0) = 0 and f(1) = 1 by definition of f.

Inductive Hypothesis:

Inductive Step:

Conclusion:

Problem 3 - Cantelli's Rabbits

Let **P(n)** be "f(n) = n" for all $\mathbf{n \geq 0}$**.**
We show **P(n)** holds for all $\mathbf{n \geq 0}$ by strong induction on **n**.

Base Cases: (n = 0, n = 1): f(0) = 0 and f(1) = 1 by definition of f.

Inductive Hypothesis: Suppose **P(0)** ∧ **P(1)** ∧ **...** ∧ **P(k)** holds for an arbitrary $\mathbf{k \geq 1}$**,**

Inductive Step:

Conclusion:

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Let **P(n)** be "f(n) = n" for all **n ≥ 0.**
We show **P(n)** holds for all **n ≥ 0** by strong induction on **n**.

Base Cases: (n = 0, n = 1): f(0) = 0 and f(1) = 1 by definition of f.

Inductive Hypothesis: Suppose **P(0) ∧ P(1) ∧ … ∧ P(k)** holds for an arbitrary **k ≥ 1,**
i.e. f(k) = k, f(k-1)=k-1, etc.

Inductive Step:

Conclusion:

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i.e. f(k) = k, f(k-1)=k-1, etc.

<u>Inductive Step:</u> Goal: Show **P(k + 1):** f(k + 1) = k + 1

<u>Conclusion:</u>

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f(k + 1) = 2 f(k) - f(k - 1) definition of f

Conclusion:

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We show **P(n)** holds for all **n ≥ 0** by strong induction on **n**.

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f(k + 1) = 2 f(k) - f(k - 1)	definition of f
= 2 (k) - (k - 1)	by I.H.

Conclusion:

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$$
\begin{aligned}
f(k+1) &= 2\,f(k) - f(k-1) && \text{definition of f} \\
&= 2\,(k) - (k-1) && \text{by I.H.} \\
&= k + 1
\end{aligned}
$$

Conclusion: Therefore, **P(n)** holds for all **n ≥ 0** by the principle of strong induction.

Midterm Review

Problem 4 - Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- soy(x) is true iff x contains soy milk.
- sugar(x) is true iff x contains sugar
- vegan(x) is true iff x is vegan.
- whole(x) is true iff x contains whole milk.
- decaf(x) is true iff x is not caffeinated.
- RobbieLikes(x) is true iff Robbie likes the drink x.

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like = and ≠.

(a) Coffee drinks with whole milk are not vegan.
(b) Robbie only likes one coffee drink, and that drink is not vegan.
(c) There is a drink that has both sugar and soy milk.

Translate the following symbolic logic statement into a (natural) English sentence. Take advantage of domain restriction.

$\forall x([\text{decaf}(x) \land \text{RobbieLikes}(x)] \rightarrow \text{sugar}(x))$

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(a) Coffee drinks with whole milk are not vegan.

$\forall x(whole(x) \rightarrow \neg\, vegan(x))$

(b) Robbie only likes one coffee drink, and that drink is not vegan.

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Let your domain of discourse be all coffee drinks.

(a) Coffee drinks with whole milk are not vegan.

$\forall x(\text{whole}(x) \rightarrow \neg\, \text{vegan}(x))$

(b) Robbie only likes one coffee drink, and that drink is not vegan.

$\exists x \forall y(\text{RobbieLikes}(x) \land \neg\, \text{Vegan}(x) \land [\text{RobbieLikes}(y) \rightarrow x = y])$

OR $\exists x(\text{RobbieLikes}(x) \land \neg\, \text{Vegan}(x) \land \forall y[\text{RobbieLikes}(y) \rightarrow x = y])$

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Let your domain of discourse be all coffee drinks.

(a) Coffee drinks with whole milk are not vegan.

$\forall x(\text{whole}(x) \rightarrow \neg\, \text{vegan}(x))$

(b) Robbie only likes one coffee drink, and that drink is not vegan.

$\exists x \forall y(\text{RobbieLikes}(x) \wedge \neg\, \text{Vegan}(x) \wedge [\text{RobbieLikes}(y) \rightarrow x = y])$

OR $\exists x(\text{RobbieLikes}(x) \wedge \neg\, \text{Vegan}(x) \wedge \forall y[\text{RobbieLikes}(y) \rightarrow x = y])$

(c) There is a drink that has both sugar and soy milk.

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(a) Coffee drinks with whole milk are not vegan.

$\forall x(\text{whole}(x) \rightarrow \neg \text{vegan}(x))$

(b) Robbie only likes one coffee drink, and that drink is not vegan.

$\exists x \forall y(\text{RobbieLikes}(x) \wedge \neg \text{Vegan}(x) \wedge [\text{RobbieLikes}(y) \rightarrow x = y])$

OR $\exists x(\text{RobbieLikes}(x) \wedge \neg \text{Vegan}(x) \wedge \forall y[\text{RobbieLikes}(y) \rightarrow x = y])$

(c) There is a drink that has both sugar and soy milk.

$\exists x(\text{sugar}(x) \wedge \text{soy}(x))$

Problem 4 - Translation

Let your domain of discourse be all coffee drinks.

Translate into English:

$\forall x([decaf(x) \wedge RobbieLikes(x)] \rightarrow sugar(x))$

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Let your domain of discourse be all coffee drinks.

Translate into English:

$\forall x([decaf(x) \wedge RobbieLikes(x)] \rightarrow sugar(x))$

“Every decaf drink that Robbie likes has sugar.”

Problem 5 - Set Theory

Suppose that $A \subseteq B$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

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Suppose $A \subseteq B$.

Let $X \in \mathcal{P}(A)$ be an arbitrary element.

Then by (REASON), $X \in \mathcal{P}(B)$.
Since X was arbitrary in $\mathcal{P}(A)$, we have shown $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

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Since $A \subseteq B$, by definition of subset again, $y \in B$.

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Since $A \subseteq B$, by definition of subset again, $y \in B$.
Since y was arbitrary in X, by definition of subset once more, $X \subseteq B$.

Then by (REASON), $X \in \mathcal{P}(B)$.
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Since $A \subseteq B$, by definition of subset again, $y \in B$.
Since y was arbitrary in X, by definition of subset once more, $X \subseteq B$.

Then by definition of powerset, $X \in \mathcal{P}(B)$.
Since X was arbitrary in $\mathcal{P}(A)$, we have shown $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Problem 6 - Number Theory

Let p be a prime number at least 3

(a) Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$. (this proof will be short!)

(b) Repeat part (a), but don't use any theorems from the Number Theory Reference Sheet. That is, show the claim directly from the definitions.

(c) From part (a), we can see that x%p can equal 1. Show that for any integer x, if $x^2 \equiv 1 \pmod{p}$, then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$. That is, show that the only value x%p can take other than 1 is $p - 1$. Hint: Suppose you have an x such that $x^2 \equiv 1 \pmod{p}$ and use the fact that $x^2 - 1 = (x - 1)(x + 1)$ Hint: You may the following theorem without proof: if p is prime and $p \mid (ab)$ then $p \mid a$ or $p \mid b$.

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Problem 6 - Number Theory

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(a) Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$.

Let y be an arbitrary integer and suppose $y \equiv 1 \pmod{p}$.

We can multiply congruences, so multiplying this congruence by itself we get
$y^2 \equiv 1^2 \pmod{p}$, or $y^2 \equiv 1 \pmod{p}$

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Since y is arbitrary, the claim holds.

Problem 6 - Number Theory

Let p be a prime number at least 3

(b) Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$

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Suppose $y \equiv 1 \pmod{p}$. By the definition of mod, $p \mid (y - 1)$.

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Let p be a prime number at least 3

(b) Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$

Suppose $y \equiv 1 \pmod{p}$. By the definition of mod, $p \mid (y - 1)$.

Therefore, by the definition of divides, there exists an integer k such that $pk = (y - 1)$

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Therefore, by the definition of divides, there exists an integer k such that $pk = (y - 1)$

By multiplying both sides of $pk = (y - 1)$ by $(y + 1)$, we get

$pk(y + 1) \quad = (y - 1)(y + 1)$

$p(k(y + 1)) = (y - 1)(y + 1)$

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Let p be a prime number at least 3

(b) Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$

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$pk(y + 1) \quad = (y - 1)(y + 1)$

$p(k(y + 1)) = (y - 1)(y + 1)$

Since $(y - 1)(y + 1) = y^2 - 1$, we have

$p(k(y + 1)) = y^2 - 1$

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$pk(y + 1) = (y - 1)(y + 1)$
$p(k(y + 1)) = (y - 1)(y + 1)$

Since $(y - 1)(y + 1) = y^2 - 1$, we have
$p(k(y + 1)) = y^2 - 1$

k and y are integers, so $(k(y + 1))$ is also an integer. By definition of divides
$p \mid y^2 - 1$. Therefore, by definition of mod, $y^2 \equiv 1 \pmod{p}$.

Problem 6 - Number Theory

Let p be a prime number at least 3

(c) Show that for any integer x, if $x^2 \equiv 1 \pmod{p}$, then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$. That is, show that the only value x%p can take other than 1 is $p - 1$.

Hint: Suppose you have an x such that $x^2 \equiv 1 \pmod{p}$ and use the fact that $x^2 - 1 = (x - 1)(x + 1)$
Hint: You may the following theorem without proof: if p is prime and $p \mid (ab)$ then $p \mid a$ or $p \mid b$.

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Suppose $x^2 \equiv 1 \pmod{p}$. By the definition of mod, $p \mid x^2 - 1$

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Problem 7 - Induction

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2$$

Prove that for all $n \in \mathbb{N}$, $S_n = (1/6)\, n(n + 1)(2n + 1)$.

Problem 7 - Induction

Prove that for all $n \in \mathbb{N}$, $S_n = (1/6)\, n(n + 1)(2n + 1)$,
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Let P(n) be the statement “” defined for all n.
We prove that P(n) is true for all n by induction on n.

<u>Base Case:</u>

<u>Inductive Hypothesis:</u>

<u>Inductive Step:</u>

<u>Conclusion:</u>

Problem 7 - Induction

Prove that for all $n \in \mathbb{N}$, $S_n = (⅙)\, n(n + 1)(2n + 1)$,
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Let P(n) be the statement "$S_n = ⅙\, n(n + 1)(2n + 1)$" defined for all $n \in \mathbb{N}$.
We prove that P(n) is true for all $n \in \mathbb{N}$ by induction on n.

Base Case:

Inductive Hypothesis:

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Prove that for all $n \in \mathbb{N}$, $S_n = (1/6)\, n(n + 1)(2n + 1)$,
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<u>Base Case:</u> P(0): When n = 0, we know the sum of the squares of the first n positive integers is the sum of no terms, so we have a sum of 0. Thus, $S_0 = 0$. Since $1/6(0)(0 + 1)((2)(0) + 1) = 0$, so we know that P(0) is true.

<u>Inductive Hypothesis:</u>

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Prove that for all $n \in \mathbb{N}$, $S_n = (1/6)\, n(n + 1)(2n + 1)$, or $1^2 + 2^2 + \cdots + n^2 = (1/6)\, n(n + 1)(2n + 1)$

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Inductive Hypothesis: Suppose that P(k) is true for some arbitrary $k \in \mathbb{N}$

Inductive Step:

Conclusion:

Problem 7 - Induction

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<u>Inductive Step:</u> Goal: Show P(k+1): $S_{k+1} = 1/6(k + 1)\, ((k + 1) + 1)\, (2(k + 1) + 1)$

<u>Conclusion:</u>

Problem 7 - Induction

Prove that for all $n \in \mathbb{N}$, $S_n = (1/6)\, n(n + 1)(2n + 1)$,
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$$S_{k+1} = 1^2 + 2^2 + \cdots + k^2 + (k + 1)^2$$

<u>Conclusion:</u>

Problem 7 - Induction

Prove that for all $n \in \mathbb{N}$, $S_n = (⅙)\, n(n + 1)(2n + 1)$,
or $1^2 + 2^2 + \cdots + n^2 = (⅙)\, n(n + 1)(2n + 1)$

Let P(n) be the statement "$S_n = ⅙\, n(n + 1)(2n + 1)$" defined for all $n \in \mathbb{N}$.
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<u>Inductive Step:</u> Goal: Show P(k+1): $S_{k+1} = ⅙(k + 1)\,((k + 1) + 1)\,(2(k + 1) + 1)$

$$S_{k+1} = 1^2 + 2^2 + \cdots + k^2 + (k + 1)^2$$
$$= S_k + (k + 1)^2$$

<u>Conclusion:</u>

Problem 7 - Induction

Prove that for all $n \in \mathbb{N}$, $S_n = (\frac{1}{6})\, n(n + 1)(2n + 1)$,
or $1^2 + 2^2 + \cdot\cdot\cdot + n^2 = (\frac{1}{6})\, n(n + 1)(2n + 1)$

Let P(n) be the statement "$S_n = \frac{1}{6}\, n(n + 1)(2n + 1)$" defined for all $n \in \mathbb{N}$.
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<u>Inductive Step:</u> Goal: Show P(k+1): $S_{k+1} = \frac{1}{6}(k + 1)\, ((k + 1) + 1)\, (2(k + 1) + 1)$

$$\begin{aligned} S_{k+1} &= 1^2 + 2^2 + \cdot\cdot\cdot + k^2 + (k + 1)^2 \\ &= S_k + (k + 1)^2 \\ &= \tfrac{1}{6}\, k(k + 1)(2k + 1) + (k+1)^2 && \text{By IH} \end{aligned}$$

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\begin{aligned}
S_{k+1} &= 1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 \\
&= S_k + (k + 1)^2 \\
&= ⅙\, k(k + 1)(2k + 1) + (k+1)^2 && \text{By IH} \\
&= (k + 1)\, (⅙\, k(2k + 1) + (k + 1))
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Problem 7 - Induction

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&= (k + 1)\,(1/6\, k(2k + 1) + (k + 1)) \\
&= 1/6(k + 1)\,(k(2k + 1) + 6(k + 1))
\end{aligned}$$

<u>Conclusion:</u>

Problem 7 - Induction

Prove that for all $n \in \mathbb{N}$, $S_n = (1/6)\, n(n+1)(2n+1)$,
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$S_{k+1} = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$
$= S_k + (k+1)^2$
$= 1/6\, k(k+1)(2k+1) + (k+1)^2$ By IH
$= (k+1)(1/6\, k(2k+1) + (k+1))$
$= 1/6(k+1)(k(2k+1) + 6(k+1))$
$= 1/6(k+1)(2k^2 + 7k + 6)$

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Problem 7 - Induction

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$$\begin{aligned} S_{k+1} &= 1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 \\ &= S_k + (k + 1)^2 \\ &= 1/6\, k(k + 1)(2k + 1) + (k+1)^2 && \text{By IH} \\ &= (k + 1)\, (1/6\, k(2k + 1) + (k + 1)) \\ &= 1/6(k + 1)\, (k(2k + 1) + 6(k + 1)) \\ &= 1/6(k + 1)\, (2k^2 + 7k + 6) \\ &= 1/6(k + 1)\, (k + 2)\, (2k + 3) \end{aligned}$$

<u>Conclusion:</u>

Problem 7 - Induction

Prove that for all $n \in \mathbb{N}$, $S_n = (1/6)\, n(n + 1)(2n + 1)$,
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&= (k + 1)\, (1/6\, k(2k + 1) + (k + 1)) \\
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\end{aligned}
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Conclusion:

Problem 7 - Induction

Prove that for all $n \in \mathbb{N}$, $S_n = (1/6)\, n(n + 1)(2n + 1)$,
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&= 1/6(k + 1)\,(k + 2)\,(2k + 3) \\
&= 1/6(k + 1)\,((k + 1) + 1)\,(2(k + 1) + 1)
\end{aligned}
$$

Thus P(k+1) holds!

<u>Conclusion:</u>

Problem 7 - Induction

Prove that for all $n \in \mathbb{N}$, $S_n = (1/6)\, n(n + 1)(2n + 1)$,
or $1^2 + 2^2 + \cdots + n^2 = (1/6)\, n(n + 1)(2n + 1)$

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&= 1/6(k + 1)\, (2k^2 + 7k + 6) \\
&= 1/6(k + 1)\, (k + 2)\, (2k + 3) \\
&= 1/6(k + 1)\, ((k + 1) + 1)\, (2(k + 1) + 1)
\end{aligned}
$$

Thus P(k+1) holds!

<u>Conclusion:</u> Therefore, P(n) holds for all integers $n \in \mathbb{N}$ by the principle of induction.

Problem 8 - Strong Induction

Robbie is planning to buy snacks for the members of his competitive roller-skating troupe. However, his local grocery store sells snacks in packs of 5 and packs of 7.

Prove that Robbie can buy exactly n snacks for all integers $n \geq 24$

Problem 8 - Strong Induction

Let P(n) be “” for all n.
We show P(n) holds for all n by strong induction on n.

Base Cases:

Inductive Hypothesis:

Inductive Step:

Conclusion:

Problem 8 - Strong Induction

Let P(n) be "Robbie can buy n snacks with packs of 5 and packs of 7 snacks" for all n ≥ 24.
We show P(n) holds for all n ≥ 24 by strong induction on n.

Base Cases:

Inductive Hypothesis:

Inductive Step:

Conclusion:

Problem 8 - Strong Induction

Let P(n) be "Robbie can buy n snacks with packs of 5 and packs of 7 snacks" for all $n \geq 24$.
We show P(n) holds for all $n \geq 24$ by strong induction on n.

Base Cases: P(24): 24 snacks can be bought with 2 packs of 7 and 2 packs of 5 snacks.
P(25): 25 snacks can be bought with 5 packs of 5 snacks.
P(26): 26 snacks can be bought with 3 packs of 7 and 1 pack of 5 snacks.
P(27): 27 snacks can be bought with 1 pack of 7 and 4 packs of 5 snacks.
P(28): 28 snacks can be bought with 4 packs of 7 snacks.

Inductive Hypothesis:

Inductive Step:

Conclusion:

Problem 8 - Strong Induction

Let $P(n)$ be "Robbie can buy n snacks with packs of 5 and packs of 7 snacks" for all $n \geq 24$.
We show $P(n)$ holds for all $n \geq 24$ by strong induction on n.

Base Cases: $P(24)$: 24 snacks can be bought with 2 packs of 7 and 2 packs of 5 snacks.
$P(25)$: 25 snacks can be bought with 5 packs of 5 snacks.
$P(26)$: 26 snacks can be bought with 3 packs of 7 and 1 pack of 5 snacks.
$P(27)$: 27 snacks can be bought with 1 pack of 7 and 4 packs of 5 snacks.
$P(28)$: 28 snacks can be bought with 4 packs of 7 snacks.

Inductive Hypothesis: Suppose that $P(24) \wedge P(25) \wedge \cdots \wedge P(k)$ is true for some arbitrary $k \geq 28$,
i.e. Robbie can buy 24 to k snacks with packs of 5 and packs of 7 snacks.

Inductive Step:

Conclusion:

Problem 8 - Strong Induction

Let P(n) be "Robbie can buy n snacks with packs of 5 and packs of 7 snacks" for all $n \geq 24$.
We show P(n) holds for all $n \geq 24$ by strong induction on n.

<u>Base Cases:</u> P(24): 24 snacks can be bought with 2 packs of 7 and 2 packs of 5 snacks.
P(25): 25 snacks can be bought with 5 packs of 5 snacks.
P(26): 26 snacks can be bought with 3 packs of 7 and 1 pack of 5 snacks.
P(27): 27 snacks can be bought with 1 pack of 7 and 4 packs of 5 snacks.
P(28): 28 snacks can be bought with 4 packs of 7 snacks.

<u>Inductive Hypothesis:</u> Suppose that $P(24) \wedge P(25) \wedge \cdots \wedge P(k)$ is true for some arbitrary $k \geq 28$,
i.e. Robbie can buy 24 to k snacks with packs of 5 and packs of 7 snacks.

<u>Inductive Step:</u> Goal: Show P(k + 1): Robbie can buy k + 1 snacks with packs of 5 and packs of 7 snacks.

<u>Conclusion:</u>

Problem 8 - Strong Induction

Let $P(n)$ be "Robbie can buy n snacks with packs of 5 and packs of 7 snacks" for all $n \geq 24$.
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<u>Inductive Hypothesis:</u> Suppose that $P(24) \land P(25) \land \cdots \land P(k)$ is true for some arbitrary $k \geq 28$,
i.e. Robbie can buy 24 to k snacks with packs of 5 and packs of 7 snacks.

<u>Inductive Step:</u> Goal: Show $P(k + 1)$: Robbie can buy $k + 1$ snacks with packs of 5 and packs of 7 snacks.

By the inductive hypothesis, we know that Robbie can buy exactly $k - 4$ snacks, so he can buy another pack of 5 to get exactly $k + 1$ snacks.

<u>Conclusion:</u>

Problem 8 - Strong Induction

Let $P(n)$ be "Robbie can buy n snacks with packs of 5 and packs of 7 snacks" for all $n \geq 24$.
We show $P(n)$ holds for all $n \geq 24$ by strong induction on n.

<u>Base Cases:</u> $P(24)$: 24 snacks can be bought with 2 packs of 7 and 2 packs of 5 snacks.
$P(25)$: 25 snacks can be bought with 5 packs of 5 snacks.
$P(26)$: 26 snacks can be bought with 3 packs of 7 and 1 pack of 5 snacks.
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<u>Inductive Hypothesis:</u> Suppose that $P(24) \land P(25) \land \cdots \land P(k)$ is true for some arbitrary $k \geq 28$, i.e. Robbie can buy 24 to k snacks with packs of 5 and packs of 7 snacks.

<u>Inductive Step:</u> Goal: Show $P(k + 1)$: Robbie can buy $k + 1$ snacks with packs of 5 and packs of 7 snacks.

By the inductive hypothesis, we know that Robbie can buy exactly $k - 4$ snacks, so he can buy another pack of 5 to get exactly $k + 1$ snacks.

<u>Conclusion:</u> Therefore, $P(n)$ holds for all $n \geq 24$ by the principle of strong induction.

That’s All, Folks!

Any questions?