Section 05: Solutions

1. GCD

(a) Calculate gcd(100, 50).

Solution:

50

(b) Calculate gcd(17, 31).

Solution:

1

(c) Find the multiplicative inverse of 6 $\pmod 7$.

Solution:

6

(d) Does 49 have an multiplicative inverse $\pmod 7$?

Solution:

It does not. Intuitively, this is because 49x for any x is going to be 0 mod 7, which means it can never be 1.

2. Extended Euclidean Algorithm

(a) Find the multiplicative inverse y of $7 \bmod 33$. That is, find y such that $7y \equiv 1 \pmod{33}$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \leq y < 33$.

Solution:

First, we find the gcd:

$$\begin{aligned}
\gcd(33,7) &= \gcd(7,5) & 33 &= \boxed{7} \bullet 4 + 5 & (1)\\
&= \gcd(5,2) & 7 &= \boxed{5} \bullet 1 + 2 & (2)\\
&= \gcd(2,1) & 5 &= \boxed{2} \bullet 2 + 1 & (3)\\
&= \gcd(1,0) & 2 &= 1 \bullet 2 + 0 & (4)\\
&= 1 & & & (5)
\end{aligned}$$

Next, we re-arrange equations (1) - (3) by solving for the remainder:

$$1 = 5 - \boxed{2} \bullet 2 \tag{6}$$
$$2 = 7 - \boxed{5} \bullet 1 \tag{7}$$
$$5 = 33 - \boxed{7} \bullet 4 \tag{8}$$
$$\tag{9}$$

Now, we backward substitute into the boxed numbers using the equations:

$$\begin{aligned} 1 &= 5 - \boxed{2} \bullet 2 \\ &= 5 - (7 - \boxed{5} \bullet 1) \bullet 2 \\ &= 3 \bullet \boxed{5} - 7 \bullet 2 \\ &= 3 \bullet (33 - \boxed{7} \bullet 4) - 7 \bullet 2 \\ &= 33 \bullet 3 + 7 \bullet -14 \end{aligned}$$

So, $1 = 33 \bullet 3 + \boxed{7} \bullet -14$. Thus, $33 - 14 = 19$ is the multiplicative inverse of 7 mod 33.

(b) Now, solve $7z \equiv 2(\text{mod } 33)$ for all of its integer solutions z.

Solution:

We already computed that 19 is the multiplicative inverse of 7 mod 33. That is, $19 \cdot 7 \equiv 1(\text{mod } 33)$.

If z is a solution to $7z \equiv 2(\text{mod } 33)$, then multiplying by 19 on both sides, we have $19 \cdot 7 \cdot z \equiv 19 \cdot 2(\text{mod } 33)$.

Substituting $19 \cdot 7 \equiv 1(\text{mod } 33)$ into this on the left gives $1 \cdot z \equiv z \equiv 19 \cdot 2 \equiv 38 \equiv 5(\text{mod } 33)$.

This shows that every solution z is congruent to 5. In other words, the set of solutions is $\{5 + 33k \mid k \in \mathbb{Z}\}$.

3. Euclid's Lemma[1]

(a) Show that if an integer p divides the product of two integers a and b, and $\gcd(p, a) = 1$, then p divides b.

Solution:

Suppose that $p \mid ab$ and $\gcd(p, a) = 1$ for integers a, b, and p. By Bezout's theorem, since $\gcd(p, a) = 1$, there exist integers r and s such that

$$rp + sa = 1.$$

Since $p \mid ab$, by the definition of divides there exists an integer k such that $pk = ab$.
By multiplying both sides of $rp + sa = 1$ by b we have,

$$\begin{aligned} rpb + s(ab) &= b \\ rpb + s(pk) &= b \\ p(rb + sk) &= b \end{aligned}$$

Since r, b, s, k are all integers, $(rb + sk)$ is also an integer. By definition we have $p \mid b$.

[1]these proofs aren't much longer than proofs you've seen so far, but it can be a little easier to get stuck – use these as a chance to practice how to get unstuck if you do!

(b) Show that if a prime p divides ab where a and b are integers, then $p \mid a$ or $p \mid b$. (Hint: Use part (a))

Solution:

Suppose that $p \mid ab$ for prime number p and integers a, b. There are two cases.

Case 1: $\text{gcd}(p, a) = 1$
In this case, $p \mid b$ by part (a).

Case 2: $\text{gcd}(p, a) \neq 1$
In this case, p and a share a common positive factor greater than 1. But since p is prime, its only positive factors are 1 and p, meaning $\text{gcd}(p, a) = p$. This says p is a factor of a, that is, $p \mid a$.

In both cases we've shown that $p \mid a$ or $p \mid b$.

4. Prime Checking

You wrote the following code, `isPrime(int n)` which you are confident returns `true` if and only if n is prime (we assume its input is always positive).

```
public boolean isPrime(int n) {
    int potentialDiv = 2;
    while (potentialDiv < n) {
        if (n % potenttialDiv == 0)
            return false;
        potentialDiv++;
    }
    return true;
}
```

Your friend suggests replacing `potentialDiv < n` with `potentialDiv <= Math.sqrt(n)`. In this problem, you'll argue the change is ok. That is, your method still produces the correct result if n is a positive integer.

We will use "nontrivial divisor" to mean a factor that isn't 1 or the number itself. Formally, a positive integer k being a "nontrivial divisor" of n means that $k|n$, $k \neq 1$ and $k \neq n$. Claim: when a positive integer n has a nontrivial divisor, it has a nontrivial divisor at most $\sqrt{n}$.

(a) Let's try to break down the claim and understand it through examples. Show an example (a specific n and k) of a nontrivial divisor, of a divisor that is not nontrivial, and of a number with only trivial divisors. **Solution:**

Some examples of "trivial" divisors: (1 of 15), (3 of 3)
Some examples of nontrivial divisors: (3 of 15), (9 of 81)
A number with only trivial divisor is just a prime number: it has no factors.

(b) Prove the claim. Hint: you may want to divide into two cases!

Solution:

Let k be a nontrivial divisor of n. Since k is a divisor, $n = kc$ for some integer c. Observe that c is also nontrivial, since if c were 1 or n then k would have to be n or 1.

We now have two cases:

Case 1: $k \leq \sqrt{n}$
If $k \leq \sqrt{n}$, then we're done because k is the desired nontrivial divisor.

Case 2: $k > \sqrt{n}$

If $k > \sqrt{n}$, then multiplying both sides by c we get $ck > c\sqrt{n}$. But $ck = n$ so $n > c\sqrt{n}$. Finally, dividing both sides by $\sqrt{n}$ gives $\sqrt{n} > c$, so c is the desired nontrivial factor.

In both cases we find a nontrivial divisor at most $\sqrt{n}$, as required.

Alternate solution (proof by contradiction): Let k be a nontrivial divisor of n. Since k is a divisor, $n = kc$ for some integer c. Observe that c is also nontrivial, since if c were 1 or n then k would have to be n or 1.

Suppose, for contradiction, that $k > \sqrt{n}$ and $c > \sqrt{n}$. Then $kc > \sqrt{n}\sqrt{n} = n$. But by assumption we have $kc = n$, so this is a contradiction. It follows that either k or c is at most $\sqrt{n}$ meaning that n has a nontrivial divisor at most $\sqrt{n}$.

(c) Informally explain why the fact about integers proved in (b) lets you change the code safely.

Solution:

The new code makes a subset of "checks" that the old code makes, thus the only concern would be that a non-prime number we found in the later checks would "slip through" without the extra checks. However, if a number has any nontrivial divisor, it will have one that is $\leq \sqrt{n}$, so even if we exit the loop early after $\sqrt{n}$ instead of n checks, our method is still guaranteed to always work.

5. Modular Arithmetic

(a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

Solution:

Suppose that $a \mid b$ and $b \mid a$, where a, b are integers. By the definition of divides, we have $a \neq 0$, $b \neq 0$ and $b = ka, a = jb$ for some integers k, j. Combining these equations, we see that $a = j(ka)$.

Then, dividing both sides by a, we get $1 = jk$. So, $\frac{1}{j} = k$. Note that j and k are integers, which is only possible if $j, k \in \{1, -1\}$. It follows that $b = -a$ or $b = a$.

(b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Solution:

Suppose $n \mid m$ with $n, m > 1$, and $a \equiv b \pmod{m}$. By definition of divides, we have $m = kn$ for some $k \in \mathbb{Z}$. By definition of congruence, we have $m \mid a - b$, which means that $a - b = mj$ for some $j \in \mathbb{Z}$. Combining the two equations, we see that $a - b = (knj) = n(kj)$. By definition of congruence, we have $a \equiv b \pmod{n}$, as required.

6. Induction with Equality

(a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Solution:

For $n \in \mathbb{N}$ let $P(n)$ be "$0 + 1 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case: We have $0 = 0 = \frac{0(0+1)}{2}$ which is $P(0)$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary integer $k \geq 0$.

Inductive Step: $\boxed{\text{Goal: Show } 0 + 1 + \cdots + (k+1) = \dfrac{(k+1)(k+2)}{2}}$.

We have

$$\begin{aligned} 0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{[Inductive Hypothesis]} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} && \text{[Factor out } (k+1)\text{]} \end{aligned}$$

This proves $P(k+1)$.

Conclusion: $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

(b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

Solution:

First, note that $\triangle_n = (0+1+2+\cdots+n)$. So, we are trying to prove $(0^3+1^3+\cdots+n^3) = (0+1+\cdots+n)^2$. Let $P(n)$ be the statement:

$$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2.$$

We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case. $0^3 = 0 = 0^2$, so $P(0)$ holds.

Inductive Hypothesis. Suppose that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$.

Inductive Step. We show $P(k+1)$:

$$\begin{aligned} 0^3 + 1^3 + \cdots (k+1)^3 &= (0^3 + 1^3 + \cdots + k^3) + (k+1)^3 && \text{[Associativity7]} \\ &= (0 + 1 + \cdots + k)^2 + (k+1)^3 && \text{[Inductive Hypothesis]} \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[Part (a)]} \\ &= (k+1)^2\left(\frac{k^2}{2^2} + (k+1)\right) && \text{[Factor } (k+1)^2\text{]} \\ &= (k+1)^2\left(\frac{k^2+4k+4}{4}\right) && \text{[Add via common denominator]} \\ &= (k+1)^2\left(\frac{(k+2)^2}{4}\right) && \text{[Factor numerator]} \\ &= \left(\frac{(k+1)(k+2)}{2}\right)^2 && \text{[Take out the square]} \\ &= (0 + 1 + \cdots + (k+1))^2 && \text{[Part (a)]} \end{aligned}$$

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of induction.