Domain Restrictions

When using quantifiers in a logical expression, domain of discourse may consist of multiple objects from different categories, e.g., cars, people, and parking lots. To narrow down our interests to a specific kind of objects, we need to restrict the domain of discourse. Here, we will walk you through how the domain of discourse can be restricted on the universal quantifier and the existential quantifier.

1. Understand the Quantifiers

1.1. $\forall x P(x)$ denotes the **universal quantification** of the atomic formula $P(x)$.

- This expression can be read as "for each $x, P(x)$ holds", "for every $x, P(x)$ holds" or "for all $x, P(x)$ holds".
- $\forall$ is the universal quantifier. $\forall x$ means every object x in the domain of discourse.

If objects in the domain of discourse can be listed, we can represent the universal quantification with **conjunctions**. For example, let the domain of discourse be $\{1, 2, ..., 10\}$ and $\mathsf{Even}(x)$ be "x is even". We have that

$$\forall x\, \mathsf{Even}(x) \equiv \mathsf{Even}(1) \land \mathsf{Even}(2) \land ... \land \mathsf{Even}(10) \equiv \mathsf{F} \land \mathsf{T} \land ... \land \mathsf{T} \equiv \mathsf{F}$$

Since $\forall x\, \mathsf{Even}(x)$ can be represented by conjunctions, it requires $\mathsf{Even}(x)$ to be true for every x in $\{1, 2, ..., 10\}$. Just one x that makes $\mathsf{Even}(x)$ become false could make the whole expression $\forall x\, \mathsf{Even}(x)$ become false. $x = 1$ here is an example. Hence, we could say that the universal quantifier is sensitive to false value (at least one false will make the whole expression become false).

1.2. $\exists x P(x)$ denotes the **existential quantification** of the atomic formula $P(x)$.

- This expression can be read as "there is an x such that $P(x)$ holds", "there exists an x such that $P(x)$ holds" or "for some $x, P(x)$ holds".
- $\exists$ is the existential quantifier. $\exists x$ means at least one object x in the domain of discourse.

If objects in the domain of discourse can be listed, we can represent the existential quantification with **disjunctions**. Again, let the domain of discourse be $\{1, 2, ..., 10\}$ and $\mathsf{Even}(x)$ be "x is even". We have that

$$\exists x\, \mathsf{Even}(x) \equiv \mathsf{Even}(1) \lor \mathsf{Even}(2) \lor ... \lor \mathsf{Even}(10) \equiv \mathsf{F} \lor \mathsf{T} \lor ... \lor \mathsf{T} \equiv \mathsf{T}$$

Since $\exists x\, \mathsf{Even}(x)$ can be represented by disjunctions, it only needs at least one x which makes $\mathsf{Even}(x)$ become true in order to have the whole expression $\exists x\, \mathsf{Even}(x)$ be evaluated to true. Look at $\mathsf{Even}(2)$ for an example. Hence, we could say that the existential quantifier is sensitive to true value (at least one true will make the whole expression become true).

2. Learn What the Domain of Discourse Can Contain

Before, we saw an example with the domain of discourse $\{1, 2, ..., 10\}$, containing only integers from one to ten. The domain of discourse acts as a universe for our interested objects and will only be defined once. Therefore, it's possible for the domain of discourse to contain objects that are not similar to each other (you will see why we need to mix objects into one domain of discourse soon when dealing with nested quantifiers). For example, $\{OrangeCat, BlackCat, BrownDog, WhiteDog, Alice, Bob\}$ consists of two cats, two dogs, and two persons.

3. Rules to Restrict the Domain

3.1. For a variable that is **universally quantified**, restrict the domain by making $\mathtt{Domain}(x)$ an extra hypothesis (or making it a hypothesis alone if there's not one already).

3.2. For a variable that is **existentially quantified**, restrict the domain by making $\mathtt{Domain}(x)$ an extra condition (i.e., just AND it in).

4. Try Restrict the Domain!

Let use the domain of discourse defined previously as $\{OrangeCat, BlackCat, BrownDog, WhiteDog, Alice, Bob\}$. Assume that

- Only dogs can bark.
- Every cat can jump.
- $\texttt{Cat}(x)$ means "x is a cat"
- $\texttt{Dog}(x)$ means "x is a dog"
- $\texttt{Human}(x)$ means "x is a human"
- $\texttt{Bark}(x)$ means "x can bark"
- $\texttt{Jump}(x)$ means "x can jump"

4.1. Let's say we want to translate **"every dog can bark"** into a predicate logic.

4.1.1. Consider the first, unrestricted-domain translation: $\forall x \, \texttt{Bark}(x)$. This is equivalent to

$$\texttt{Bark}(OrangeCat) \land \texttt{Bark}(BlackCat) \land \texttt{Bark}(BrownDog) \land \texttt{Bark}(WhiteDog) \land \texttt{Bark}(Alice) \land \texttt{Bark}(Bob)$$

which can be evaluated to

$$\mathsf{F} \land \mathsf{F} \land \mathsf{T} \land \mathsf{T} \land \mathsf{F} \land \mathsf{F} \equiv \mathsf{F}$$

This is <u>not</u> what we wanted because it is true that every dog can bark. Remember that the universal quantifier is sensitive to false value? We applied Bark to every object in the domain of discourse without focusing on just dogs! Instead, let's try using the restriction rule by making our restriction be a hypothesis (since we are dealing with the universal quantifier).

4.1.2. Consider the restricted-domain translation: $\forall x (\texttt{Dog}(x) \to \texttt{Bark}(x))$. This is equivalent to

$$[\mathsf{F} \to \texttt{Bark}(OrangeCat)] \land [\mathsf{F} \to \texttt{Bark}(BlackCat)] \land [\mathsf{T} \to \texttt{Bark}(BrownDog)] \land$$

$$[\mathsf{T} \to \texttt{Bark}(WhiteDog)] \land [\mathsf{F} \to \texttt{Bark}(Alice)] \land [\mathsf{F} \to \texttt{Bark}(Bob)]$$

which can be avaluated to

$$\mathsf{T} \land \mathsf{T} \land \mathsf{T} \land \mathsf{T} \land \mathsf{T} \land \mathsf{T} \equiv \mathsf{T}$$

By having the restricting condition $\texttt{Dog}(x)$ be a hypothesis, any input x that is not a dog will not be considered because $(\texttt{Dog}(x) \to \texttt{Bark}(x))$ will give us a vacuous truth and it will not affect the truth value of the whole statement $\forall x(\texttt{Dog}(x) \to \texttt{Bark}(x))$. Remember that the universal quantifier is sensitive to false value? I grayed out non-dog input to help you see them better.

Now, imagine you try to do $\forall x(\texttt{Dog}(x) \land \texttt{Bark}(x))$. Why won't it work? (again, the universal quantifier is sensitive to false value due to logical connective AND).

4.2. Let's try some existential quantifier by translating **"some cat cannot jump"** into a predicate logic.

4.2.1. Consider an unrestricted-domain translation: $\exists x(\neg \texttt{Jump}(x))$. This is equivalent to

$$\neg \texttt{Jump}(OrangeCat) \lor \neg \texttt{Jump}(BlackCat) \lor \neg \texttt{Jump}(BrownDog) \lor \neg \texttt{Jump}(WhiteDog) \lor \neg \texttt{Jump}(Alice) \lor \neg \texttt{Jump}(Bob)$$

which can be evaluated to

$$\mathsf{F} \lor \mathsf{F} \lor \mathsf{T} \lor \mathsf{T} \lor \mathsf{T} \lor \mathsf{T} \equiv \mathsf{T}$$

There's an <u>error</u> here. "Some cat cannot jump" should be evaluated to false since every cat can jump. The existential quantifier is sensitive to true value, and we should avoid considering those input that are not cats because it affected our statement. We can try using the restriction rule by ANDing $\neg \texttt{Jump}(x)$ with $Cat(x)$ to only consider cats from the domain.

4.2.2. Consider the restricted-domain translation: $\exists x(\texttt{Cat}(x) \land \neg\,\texttt{Jump}(x))$. This is equivalent to

$$[\mathsf{T} \land \neg\,\texttt{Jump}(OrangeCat)] \lor [\mathsf{T} \land \neg\,\texttt{Jump}(BlackCat)] \lor [\mathsf{F} \land \neg\,\texttt{Jump}(BrownDog)] \lor$$

$$[\mathsf{F} \land \neg\,\texttt{Jump}(WhiteDog)] \lor [\mathsf{F} \land \neg\,\texttt{Jump}(Alice)] \lor [\mathsf{F} \land \neg\,\texttt{Jump}(Bob)]$$

which can be evaluated to

$$\mathsf{F} \lor \mathsf{F} \lor \mathsf{F} \lor \mathsf{F} \lor \mathsf{F} \lor \mathsf{F} \equiv \mathsf{F}$$

You can see that when x is $BrownDog, WhiteDog, Alice$ or Bob, it will not affect the whole statement at all since it is not a cat and $(\texttt{Cat}(x) \land \neg\,\texttt{Jump}(x))$ will just be evaluated to false. Then, we could just focus on the values produced by cats.

Let's try $\exists x(\texttt{Cat}(x) \rightarrow \neg\,\texttt{Jump}(x))$. Why won't it work? (again, the existential quantifier is sensitive to true value due to logical connective OR).

5. What About Nested Quantifiers?

I believe that you, now, agree on the rules such that

- To restrict the domain under a $\forall$ quantifier, add a hypothesis to an implication.
- To restrict the domain under an $\exists$ quantifier, AND in the restriction.

Let's use the same domain of discourse and the same assumptions as before. Let's also add new assumptions:

- `OrangeCat` only loves `Alice` and `Bob`
- `BlackCat` only loves `Alice`
- Both `BrownDog` and `WhiteDog` do not love anyone
- $\texttt{Loves}(x, y)$ means "x loves y"
- $\texttt{Orange}(x)$ menas "x is orange"

5.1. Let's say we want to translate **"every cat loves some human"** into a predicate logic.

Consider an unrestricted-domain translation: $\forall x \exists y\, \texttt{Loves}(x, y)$. This is equivalent to

$$\big[\texttt{Loves}(\texttt{OrangeCat}, \texttt{OrangeCat}) \lor \texttt{Loves}(\texttt{OrangeCat}, \texttt{BlackCat}) \lor \texttt{Loves}(\texttt{OrangeCat}, \texttt{BrownDog}) \lor \ldots \lor \texttt{Loves}(\texttt{OrangeCat}, \texttt{Bob})\big] \land$$

$$\big[\texttt{Loves}(\texttt{BlackCat}, \texttt{OrangeCat}) \lor \texttt{Loves}(\texttt{BlackCat}, \texttt{BlackCat}) \lor \texttt{Loves}(\texttt{BlackCat}, \texttt{BrownDog}) \lor \ldots \lor \texttt{Loves}(\texttt{BlackCat}, \texttt{Bob})\big] \land$$

$$\big[\texttt{Loves}(\texttt{BrownDog}, \texttt{OrangeCat}) \lor \texttt{Loves}(\texttt{BrownDog}, \texttt{BlackCat}) \lor \texttt{Loves}(\texttt{BrownDog}, \texttt{BrownDog}) \lor \ldots \lor \texttt{Loves}(\texttt{BrownDog}, \texttt{Bob})\big] \land$$

$$\ldots$$

$$\big[\texttt{Loves}(\texttt{Bob}, \texttt{OrangeCat}) \lor \texttt{Loves}(\texttt{Bob}, \texttt{BlackCat}) \lor \texttt{Loves}(\texttt{Bob}, \texttt{BrownDog}) \lor \ldots \lor \texttt{Loves}(\texttt{Bob}, \texttt{Bob})\big]$$

This expression will be evaluated to false because we know that both `BrownDog` and `WhiteDog` do not love anyone. However, the statement "every cat loves some human" should be true based on our assumption. We had some error here, and the error happened because we didn't narrow down our interests to just "every cat" and "some human". Here is what we want the statement to look like (notice that we "for all" (conjunctions) on cat and "for some" (disjunctions) on human):

$$\big[\texttt{Loves}(\texttt{OrangeCat}, \texttt{Alice}) \lor \texttt{Loves}(\texttt{OrangeCat}, \texttt{Bob})\big] \land \big[\texttt{Loves}(\texttt{BlackCat}, \texttt{Alice}) \lor \texttt{Loves}(\texttt{BlackCat}, \texttt{Bob})\big]$$

To restrict the domain on the variable x to be just cats, we make a new implication with a hypothesis $\texttt{Cat}(x)$. Then, we have $\forall x \exists y(\texttt{Cat}(x) \rightarrow \texttt{Loves}(x, y))$. However, it's done yet because the variable y still refers to any object in the domain, but we want y to just be human. To achieve that, we AND the restriction $\texttt{Human}(y)$ on the conclusion to narrow down our focus in disjunctions to just human. Finally, we have the correct translation: $\forall x \exists y(\texttt{Cat}(x) \rightarrow (\texttt{Human}(y) \land \texttt{Loves}(x, y)))$ which is equivalent to $\forall x(\texttt{Cat}(x) \rightarrow \exists y(\texttt{Human}(y) \land \texttt{Loves}(x, y)))$

5.2. Now, let's try translating **"there is an orange cat who loves every human"** into a predicate logic.

First, we know that $\exists x \forall y(\texttt{Loves}(x, y))$ would definitely not work here because we didn't restrict anything inside the domain. Looking at the statement, here is what we may want to achieve at the end:

$$\left[\texttt{Loves}(\texttt{OrangeCat}, \texttt{Alice}) \;\wedge\; \texttt{Loves}(\texttt{OrangeCat}, \texttt{Bob})\right] \vee \texttt{F} \vee \texttt{F} \vee \texttt{F} \vee \texttt{F} \vee \texttt{F}$$

Note that $\texttt{F}$'s came from $BlackCat, BrownDog, WhiteDog, Alice,$ and Bob which we do not want to consider since they are not an orange cat.

Since x is a variable that is existentially quantified, we can add $(\texttt{Cat}(x) \wedge \texttt{Orange}(x))$ as our condition to narrow down the domain to just an orange cat. Then, we have $\exists x \forall y((\texttt{Cat}(x) \wedge \texttt{Orange}(x)) \wedge \texttt{Loves}(x, y))$. However, this is not finished just yet because we haven't restrict the variable y to focus on just human. Since y is universally quantified, let's make $\texttt{Human}(y)$ as a hypothesis. Because we have x as an outer scope (think of it as a outer loop) and y as an inner scope (or an inner loop), we will apply the restriction $\texttt{Human}(y)$ only when we focus on the orange cat. In other words, if we know that x is not an orange cat, we don't have to worry about y anyway since $(\texttt{Cat}(x) \wedge \texttt{Orange}(x))$ will just evaluate to false. Finally, we will have a correct translation as $\exists x \forall y((\texttt{Cat}(x) \wedge \texttt{Orange}(x)) \wedge (\texttt{Human}(y) \rightarrow \texttt{Loves}(x, y)))$ which is equivalent to $\exists x((\texttt{Cat}(x) \wedge \texttt{Orange}(x)) \wedge \forall y(\texttt{Human}(y) \rightarrow \texttt{Loves}(x, y)))$.