Let’s Try Another! Stamp Collecting

I have 4 cent stamps and 5 cent stamps (as many as I want of each). Prove that I can make exactly \( n \) cents worth of stamps for all \( n \geq 12 \).

Try for a few values.
Then think...how would the inductive step go?
Define \( P(n) \) I can make \( n \) cents of stamps with just 4 and 5 cent stamps. We prove \( P(n) \) is true for all \( n \geq 12 \) by induction on \( n \).

Base Case:
12 cents can be made with three 4 cent stamps.

Inductive Hypothesis Suppose \( P(k), k \geq 12 \).

Inductive Step:
We want to make \( k + 1 \) cents of stamps. By IH we can make \( k \) cents exactly with stamps. Replace one of the 4 cent stamps with a 5 cent stamp.

\( P(n) \) holds for all \( n \) by the principle of induction.
Stamp Collection, Done Wrong

What if the starting point doesn’t have any 4 cent stamps?
Like, say, 15 cents = 5+5+5.
Gridding

I’ve got a bunch of these 3 piece tiles. I want to fill a $2^n \times 2^n$ grid ($n \geq 1$) with the pieces, except for a 1x1 spot in a corner.
Gridding: Not a formal proof, just a sketch

Base Case: \( n = 1 \)

Inductive hypothesis: Suppose you can tile a \( 2^k \times 2^k \) grid, except for a corner.

Inductive step: \( 2^{k+1} \times 2^{k+1} \), divide into quarters. By IH can tile...
Recursively Defined Functions

Just like induction works will with recursive code, it also works well for recursively-defined functions.

Define the Fibonacci numbers as follows:

\[
\begin{align*}
    f(0) &= 1 \\
    f(1) &= 1 \\
    f(n) &= f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.
\end{align*}
\]

*This is a somewhat unusual definition, \( f(0) = 0, f(1) = 1 \) is more common.*
Fibonacci Inequality

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

\[ f(0) = 1; \quad f(1) = 1 \]
\[ f(n) = f(n - 1) + f(n - 2) \text{ for all } n \in \mathbb{N}, n \geq 2. \]
Fibonacci Inequality

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

Define $P(n)$ to be "$f(n) \leq 2^n$". We show $P(n)$ is true for all $n \geq 0$ by induction on $n$.

Base Cases: ($n = 0$): $f(0) = 1 \leq 1 = 2^0$.
($n = 1$): $f(1) = 1 \leq 2 = 2^1$.

Inductive Hypothesis: Suppose $P(0) \land P(1) \land \cdots \land P(k)$ for an arbitrary $k \geq 1$.

Inductive step:

Target: $P(k + 1)$, i.e. $f(k + 1) \leq 2^{k+1}$.

$f(0) = 1$; $f(1) = 1$; $f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$. 

$f(k+1) = f(k) + f(k-1)$ by defn of $f$.

$\leq 2^k + 2^{k-1}$ by IH twice.

$\leq 2^{k+1}$

$16 = 2^4$.
Fibonacci Inequality

Show that \( f(n) \leq 2^n \) for all \( n \geq 0 \) by induction.

Define \( P(n) \) to be “\( f(n) \leq 2^n \)” We show \( P(n) \) is true for all \( n \geq 0 \) by induction on \( n \).

Base Cases: \( (n = 0) \): \( f(0) = 1 \leq 1 = 2^0 \).

\( (n = 1) \): \( f(1) = 1 \leq 2 = 2^1 \).

Inductive Hypothesis: Suppose \( P(0) \land P(1) \land \cdots \land P(k) \) for an arbitrary \( k \geq 1 \).

Inductive step: \( f(k + 1) = f(k) + f(k - 1) \) by the definition of the Fibonacci numbers. Applying IH twice, we have \( f(k + 1) \leq 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1} \).

Therefore, we have \( P(n) \) for all \( n \geq 0 \) by the principle of induction.
Claim: $3 | (2^{2n} - 1)$ for all $n \in \mathbb{N}$.

[Define $P(n)$]

Base Case
Inductive Hypothesis
Inductive Step

[conclusion]
Claim: $3|(2^{2n} - 1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$3|(2^{2n} - 1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and $0$ is an integer, $3|(2^{2\cdot0} - 1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step:

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)} - 1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.
Claim: $3|(2^{2n} - 1)$ for all $n \in \mathbb{N}$.

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Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Inductive Step: By inductive hypothesis, $3|(2^{2k} - 1)$. i.e. there is an integer $j$ such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1$

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)} - 1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.
Claim: $3|(2^{2n} - 1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$3|(2^{2n} - 1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2 \cdot 0} - 1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k} - 1)$. i.e. there is an integer $j$ such that $3j = 2^{2k} - 1$.

$$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 4(2^{2k} - 1) + 4 - 1$$

By IH, we can replace $2^{2k} - 1$ with $3j$ for an integer $j$

$$2^{2(k+1)} - 1 = 4(3j) + 4 - 1 = 3(4j) + 3 = 3(4j + 1)$$

Since $4j + 1$ is an integer, we meet the definition of divides and we have:

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)} - 1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.
Claim: $3|(2^{2^n} - 1)$ for all $n \in \mathbb{N}$.

That inductive step might still seem like magic.

It sometimes helps to run through examples, and look for patterns:

$2^{2^0} - 1 = 0 = 3 \cdot 0$
$2^{2^1} - 1 = 3 = 3 \cdot 1$
$2^{2^2} - 1 = 15 = 3 \cdot 5$
$2^{2^3} - 1 = 63 = 3 \cdot 21$
$2^{2^4} - 1 = 255 = 3 \cdot 85$
$2^{2^5} - 1 = 1023 = 3 \cdot 341$

The divisor goes from $k$ to $4k + 1$
- $0 \rightarrow 4 \cdot 0 + 1 = 1$
- $1 \rightarrow 4 \cdot 1 + 1 = 5$
- $5 \rightarrow 4 \cdot 5 + 1 = 21$

... That might give us a hint that $4k + 1$ will be in the algebra somewhere, and give us another intermediate target.
Induction: Hats!

You have \( n \) people in a line \((n \geq 2)\). Each of them wears either a purple hat or a gold hat. The person at the front of the line wears a purple hat. The person at the back of the line wears a gold hat.

Show that for every arrangement of the line satisfying the rule above, there is a person with a purple hat next to someone with a gold hat.

Yes this is kinda obvious. I promise this is good induction practice.

Yes you could argue this by contradiction. I promise this is good induction practice.
Induction: Hats!

Define $P(n)$ to be “in every line of $n$ people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat”

We show $P(n)$ for all integers $n \geq 2$ by induction on $n$.

Base Case: $n = 2$

Inductive Hypothesis:

Inductive Step:

By the principle of induction, we have $P(n)$ for all $n \geq 2$
Induction: Hats!

Define $P(n)$ to be “in every line of $n$ people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat”

We show $P(n)$ for all integers $n \geq 2$ by induction on $n$.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Target: there is someone in a purple hat next to someone in a gold hat.

By the principle of induction, we have $P(n)$ for all $n \geq 2$
Induction: Hats!

Define $P(n)$ to be “in every line of $n$ people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat”

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Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Case 1: There is someone with a purple hat next to the person in the gold hat at one end. Then those people are the required adjacent opposite hats.

Case 2: There is a person with a gold hat next to the person in the gold hat at the end. Then the line from the second person to the end is length $k$, has a gold hat at one end and a purple hat at the other. Applying the inductive hypothesis, there is an adjacent, opposite-hat wearing pair.

In either case we have $P(k + 1)$.

By the principle of induction, we have $P(n)$ for all $n \geq 2$
Fibonacci Inequality Two

Show that \( f(n) \geq 2^{n/2} \) for all \( n \geq 2 \) by induction.

[Define \( P(n) \)]

Base Cases:

Inductive Hypothesis:

Inductive step:

Therefore, we have \( P(n) \) for all \( n \geq 0 \) by the principle of induction.
Show that \( f(n) \geq 2^{n/2} \) for all \( n \geq 2 \) by induction.

Define \( P(n) \) to be "\( f(n) \geq 2^{n/2} \)". We show \( P(n) \) is true for all \( n \geq 2 \) by induction on \( n \).

Base Cases: \( f(2) = f(1) + f(0) = 2 \geq 2 = 2^{1} = 2^{2/2} \)

\[
f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}
\]

Inductive Hypothesis: Suppose \( P(2) \land P(3) \land \cdots \land P(k) \) for an arbitrary \( k \geq 3 \).

Inductive step: \( f(k + 1) = f(k) + f(k - 1) \) by the definition of the Fibonacci numbers. Applying IH twice, we have

\[
\text{Target: } f(k + 1) \geq 2^{(k+1)/2}
\]

Therefore, we have \( P(n) \) for all \( n \geq 0 \) by the principle of induction.
Fibonacci Inequality Two

Show that \( f(n) \geq 2^{n/2} \) for all \( n \geq 2 \) by induction.

Define \( P(n) \) to be "\( f(n) \geq 2^{n/2} \)." We show \( P(n) \) is true for all \( n \geq 2 \) by induction on \( n \).

Base Cases: \( f(2) = f(1) + f(0) = 2 \geq 2 = 2^{1} = 2^{2/2} \)

\[ f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2} \]

Inductive Hypothesis: Suppose \( P(2) \land P(3) \land \cdots \land P(k) \) for an arbitrary \( k \geq 3 \).

Inductive step: \( f(k + 1) = f(k) + f(k - 1) \) by the definition of the Fibonacci numbers. Applying IH twice, we have

\[ f(k + 1) \geq 2^{k/2} + 2^{(k-1)/2} \]

\[ \geq 2^{(k+1)/2} \]

Therefore, we have \( P(n) \) for all \( n \geq 0 \) by the principle of induction.

\[ f(0) = 1; \quad f(1) = 1 \]

\[ f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2. \]
Fibonacci Inequality Two

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$". We show $P(n)$ is true for all $n \geq 2$ by induction on $n$.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

\[ f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2} \]

Inductive Hypothesis: Suppose $P(2) \land P(3) \land \cdots \land P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

\[
 f(k+1) \geq 2^{k/2} + 2^{(k-1)/2} \\
 = 2^{(k-1)/2} (\sqrt{2} + 1) \\
 \geq 2^{(k-1)/2} \cdot 2 \\
 \geq 2^{(k+1)/2}
\]

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.
More Practice
Even More Induction Practice

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Claim: $h(n) \geq g(n)$ for all integers $n \geq 1$
Define $P(n)$ to be "$h(n) \geq g(n)$ for all integers $n \geq 1$"

We show $P(n)$ for all $n \geq 1$ by induction on $n$.

Base Case

Inductive Hypothesis:

Inductive Step:

Thus $P(k + 1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on $n$.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$
Define $P(n)$ to be "$h(n) \geq g(n)$ for all integers $n \geq 1$"

We show $P(n)$ for all $n \geq 1$ by induction on $n$.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$g(k + 1) = (k + 1) \cdot g(k)$

$= (k + 1)^{k+1}$.

Thus $P(k + 1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on $n$.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$
Define $P(n)$ to be "$h(n) \geq g(n)$ for all integers $n \geq 1$"

We show $P(n)$ for all $n \geq 1$ by induction on $n$.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$g(k + 1) = (k + 1) \cdot g(k)$

$\leq (k + 1) \cdot h(k)$ by IH.

$= (k + 1)^{k+1}$.

Thus $P(k + 1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on $n$.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$
Define $P(n)$ to be "$h(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on $n$.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$g(k + 1) = (k + 1) \cdot g(k)$

$\leq (k + 1) \cdot h(k)$ \hspace{10pt} \text{by IH.}$

$\leq (k + 1) \cdot k^k$ \hspace{10pt} \text{by definition of } h(k)$

$\leq (k + 1) \cdot (k + 1)^k$

$= (k + 1)^{k+1}$.

Thus $P(k + 1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on $n$.
Define $P(n)$ to be "$h(n) \geq g(n)$ for all integers $n \geq 1$\n
We show $P(n)$ for all $n \geq 1$ by induction on $n$.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step: $g(k + 1) = (k + 1) \cdot g(k)$

$\leq (k + 1) \cdot h(k)$ by IH.

$\leq (k + 1) \cdot k^k$ by definition of $h(k)$

$\leq (k + 1) \cdot (k + 1)^k$

$= (k + 1)^{k+1}$.

Thus $P(k + 1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on $n$.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$
Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on $n$.

Base Case ($n = 0$):

Inductive Hypothesis:

Inductive Step:

[Conclusion]
Let \( P(n) \) be \( \sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2} \)

Show \( P(n) \) for all \( n \in \mathbb{N} \) by induction on \( n \).

Base Case (\( n = 0 \)): \( \sum_{i=0}^{0} 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot0+4)}{2} \)

Inductive Hypothesis: Suppose \( P(k) \) is true for an arbitrary \( k \geq 0 \).

Inductive Step:

\[
\sum_{i=0}^{k+1} 2 + 3i = \frac{([k+1]+1)(3[k+1]+4)}{2}
\]
Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on $n$.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step: $\sum_{i=0}^{k+1} 2 + 3i = (\sum_{i=0}^{k} 2 + 3i) + (2 + 3(k + 1))$. By IH, we have:

$\sum_{i=0}^{k+1} 2 + 3i = \frac{(k+1)(3k+4)}{2} + 2 + 3k + 3 = \frac{([k + 1] + 1)(3[k + 1] + 4)}{2}$
Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on $n$.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

$\sum_{i=0}^{k+1} 2 + 3i = (\sum_{i=0}^{k} 2 + 3i) + (2 + 3(k + 1))$. By IH, we have:

$\sum_{i=0}^{k+1} 2 + 3i = \frac{(k+1)(3k+4)}{2} + 2 + 3k + 3 = \frac{3k^2+7k+4}{2} + \frac{6k+10}{2} = \frac{3k^2+13k+14}{2} = \frac{(3k+7)(k+2)}{2}$

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by induction on $n$. 