Section 09 (Part B): Solutions

1. Reflexivity Proof

Let R and S be relations on a set A. Consider the following claim:

Given that R and S are reflexive, it follows that $R \cup S$ is reflexive.

(a) What does "$R \cup S$ is reflexive" become if we unroll the definition of reflexive? **Solution:**

$\forall x\, ((x, x) \in R \cup S)$

(b) Write a formal proof that the claim from (a) holds. **Solution:**

1.	$\forall x\, ((x, x) \in R)$	Given
2.	$\forall x\, ((x, x) \in S)$	Given
3.	Let x be an arbitrary object.	
4.	$(x, x) \in R$	Elim $\forall$: 1
5.	$((x, x) \in R) \vee ((x, x) \in S)$	Intro $\vee$: 4
6.	$((x, x) \in R \cup S)$	Def of Union: 5
7.	$\forall x\, ((x, x) \in R \cup S)$	Intro $\forall$: 3, 6

(c) Translate the formal proof into an English proof of the original claim. **Solution:**

Let x be arbitrary. Since we were given that R is reflexive, we know that $(x, x) \in R$. Then, it must be the case that $(x, x) \in R$ or $(x, x) \in S$. By definition of union, we know then that $(x, x) \in R \cup S$. Since x was arbitrary, we have shown that $R \cup S$ is reflexive.

2. Relations

Suppose that R is reflexive. Prove that $R \subseteq R^2$, with a formal proof and then an English proof. You may use the following definition:

$$R^2 := R \circ R$$

Solution:

Formal proof:

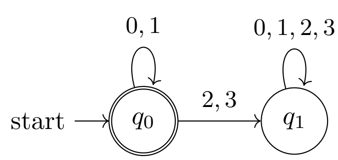
1.	$\forall x\,((x, x) \in R)$	Given
2.	Let x and y be arbitrary.	
3.1.	$(x, y) \in R$	Assumption
3.2.	$(y, y) \in R$	Elim $\forall$: 1
3.3.	$(x, y) \in R \wedge (y, y) \in R$	Intro $\wedge$: 3.1, 3.2
3.4.	$\exists z((x, z) \in R \wedge (z, y) \in R)$	Intro $\exists$: 3.3
3.5.	$(x, y) \in R \circ R$	Def of Composition: 3.4
3.6.	$(x, y) \in R^2$	Def of R^2: 3.5
3.	$(x, y) \in R \to (x, y) \in R^2$	Direct Proof
4.	$\forall x\,\forall y\,((x, y) \in R \to (x, y) \in R^2)$	Intro $\forall$: 2, 3
5.	$R \subseteq R^2$	Def of $\subseteq$: 4

English proof: Let x and y be arbitrary. Suppose $(x, y) \in R$. Since we were given that R is reflexive, we know $(y, y) \in R$ as well. Since there is a z such that $(x, z) \in R$ and $(z, y) \in R$, it follows that $(x, y) \in R^2$. Thus, since x and y were arbitrary, by definition of subset$R \subseteq R^2$.

3. DFAs, Stage 1

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1, 2, 3\}$.

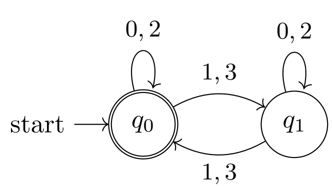
(a) All binary strings.
Solution:



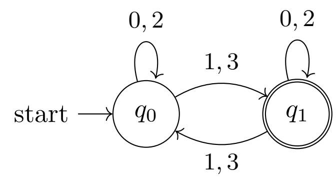
q_0**:** binary strings

q_1**:** strings that contain a character which is not 0 or 1.

(b) All strings whose digits sum to an even number.
Solution:



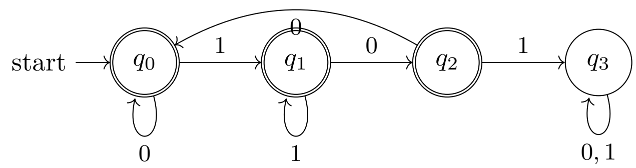
(c) All strings whose digits sum to an odd number.
Solution:



4. DFAs, Stage 2

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1\}$.

(a) All strings which do not contain the substring 101.
Solution:



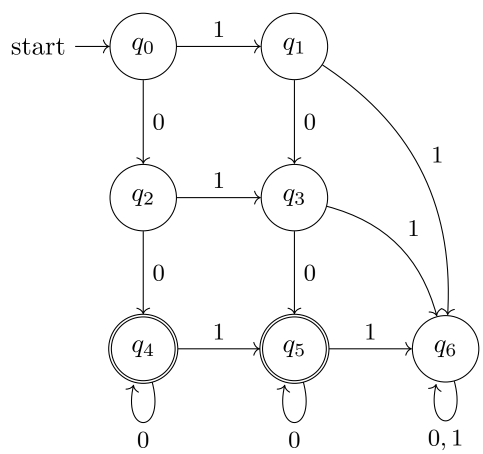
q_3: string that contain 101.

q_2: strings that don't contain 101 and end in 10.

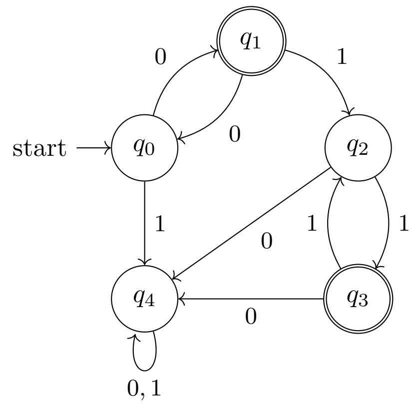
q_1: strings that don't contain 101 and end in 1.

q_0: ε, 0, strings that don't contain 101 and end in 00.

(b) All strings containing at least two 0's and at most one 1.
Solution:

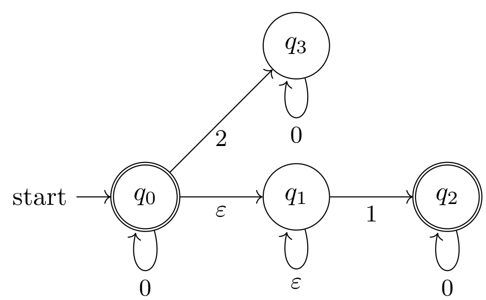


(c) All strings containing an even number of 1's and an odd number of 0's and not containing the substring 10.
Solution:



5. NFAs

(a) What language does the following NFA accept?

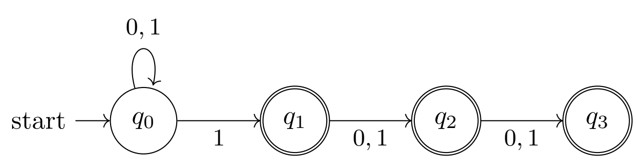


Solution:

All strings of only 0's and 1's not containing more than one 1.

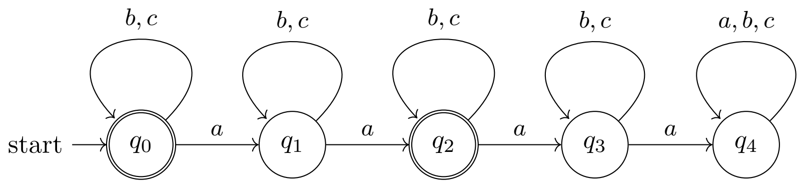
(b) Create an NFA for the language "all binary strings that have a 1 as one of the last three digits".
Solution:

The following is one such NFA:



6. DFA Minimization

(a) Minimize the following DFA:



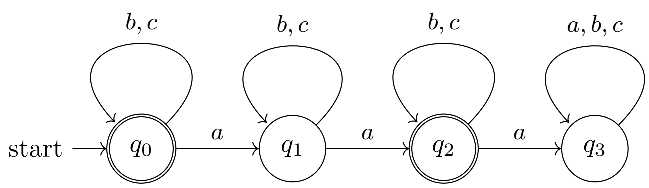
Solution:

Step 1: q_0, q_2 are final states and the rest are not final. So, we start with the initial partition with the following groups: group 1 is $\{q_0, q_2\}$ and group 2 is $\{q_1, q_3, q_4\}$.

Step 2: q_1 is sending a to group 1 while q_3, q_4 are sending a to group 2. So, we divide group 2. We get the following groups: group 1 is $\{q_0, q_2\}$, group 3 is $\{q_1\}$ and group 4 is $\{q_3, q_4\}$.

Step 3: q_0 is sending a to group 3 and q_2 is sending a to group 4. So, we divide group 1. We will have the following groups: group 3 is $\{q_1\}$, group 4 is $\{q_3, q_4\}$, group 5 is $\{q_0\}$ and group 6 is $\{q_2\}$.

The minimized DFA is the following:



(b) Minimize your solution to problem 1.

Solution:

Step 1: $\{q_0, q_1\}$ and $\{q_2\}$ are final states and the rest are not final. So, we start with the initial partition with the following groups: group 1 is $\{\{q_0, q_1\}, \{q_2\}\}$ and group 2 is $\{\{q_3\}, \emptyset\}$.

Step 2: $\{q_0, q_1\}$ sends 1 to group 1 while $\{q_2\}$ sends 1 to group 2. So, we divide group 1. We get the following groups: group 1 is now just $\{\{q_0, q_1\}\}$, group 2 is $\{\{q_3\}, \emptyset\}$, and the new group 3 is $\{\{q_2\}\}$.

Collapsing group 2 into the single state $\emptyset$ gives us this minimized DFA: