Section 09 (Part A): Solutions

1. CFGs

(a) All binary strings that end in 00.
Solution:

$$\mathbf{S} \to 0\mathbf{S} \mid 1\mathbf{S} \mid 00$$

(b) All binary strings that contain at least three 1's.
Solution:

$$\mathbf{S} \to \mathbf{TTT}$$
$$\mathbf{T} \to 0\mathbf{T} \mid \mathbf{T}0 \mid 1\mathbf{T} \mid 1$$

(c) All strings over {0,1,2} with the same number of 1s and 0s and exactly one 2.
Hint: Try modifying the grammar from lecture for binary strings with the same number of 1s and 0s. (You may need to introduce new variables in the process.)
Solution:

We can do this by slightly modifying the grammar from lecture.

$$\mathbf{S} \to 2\mathbf{T} \mid \mathbf{T}2 \mid \mathbf{ST} \mid \mathbf{TS} \mid 0\mathbf{S}1 \mid 1\mathbf{S}0$$
$$\mathbf{T} \to \mathbf{TT} \mid 0\mathbf{T}1 \mid 1\mathbf{T}0 \mid \varepsilon$$

T is the grammar from lecture. It generates all binary strings with the same number of 1s and 0s.

S matches a 2 at the beginning or end. The rest of the string must then match **T** since it cannot have another 2. If neither the first nor last character is a 2, then it falls into the usual cases for matching 0s and 1s, so we can mostly use the same rules as **T**. The main change is that **SS** becomes **ST** | **TS** to ensure that exactly one of the two parts contains a 2. The other change is that there is no ε since a 2 must appear somewhere.

2. Good, Good, Good, Good Relations

In each part of this problem, we define a relation R on this set. For each one, state whether R is or is not reflexive, symmetric, antisymmetric, and/or transitive. If a relation does not have a property, state a counterexample.

(a) Consider the relation $R = \{(x, y) : x = y + 1\}$ on $\mathbb{N}$. **Solution:**

not reflexive (counterexample: $(1, 1) \notin R$), not symmetric (counterexample: $(2, 1) \in R$ but $(1, 2) \notin R$), antisymmetric, not transitive (counterexample: $(3, 2) \in R$ and $(2, 1) \in R$, but $(3, 1) \notin R$

(b) Consider the relation $R = \{(x, y) : x^2 = y^2\}$ on $\mathbb{R}$.
Solution:

reflexive, symmetric, not antisymmetric (counterexample: $(-2, 2) \in R$ and $(2, -2) \in R$ but $2 \neq -2$), transitive