

Section 07: Midterm Review and Strong Induction

1. Midterm Review: Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- $\text{soy}(x)$ is true iff x contains soy milk.
- $\text{whole}(x)$ is true iff x contains whole milk.
- $\text{sugar}(x)$ is true iff x contains sugar
- $\text{decaf}(x)$ is true iff x is not caffeinated.
- $\text{vegan}(x)$ is true iff x is vegan.
- $\text{RobbieLikes}(x)$ is true iff Robbie likes the drink x .

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like $=$ and $\neq$.

- Coffee drinks with whole milk are not vegan.
- Robbie only likes one coffee drink, and that drink is not vegan.
- There is a drink that has both sugar and soy milk.

Translate the following symbolic logic statement into a (natural) English sentence. Take advantage of domain restriction.

$$\forall x([\text{decaf}(x) \wedge \text{RobbieLikes}(x)] \rightarrow \text{sugar}(x))$$

2. Midterm Review: Number Theory

Let p be a prime number at least 3, and let x be an integer such that $x^2 \bmod p = 1$.

- Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$. (this proof will be short!)
(Try to do this without using the theorem "Raising Congruences To A Power")
- Repeat part (a), but don't use any theorems from the Number Theory Reference Sheet. That is, show the claim directly from the definitions.
- From part (a), we can see that $x \bmod p$ can equal 1. Show that for any integer x , if $x^2 \equiv 1 \pmod{p}$, then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$. That is, show that the only value $x \bmod p$ can take other than 1 is $p-1$.
Hint: Suppose you have an x such that $x^2 \equiv 1 \pmod{p}$ and use the fact that $x^2 - 1 = (x-1)(x+1)$
Hint: You may use the following theorem without proof: if p is prime and $p \mid (ab)$ then $p \mid a$ or $p \mid b$.

3. Midterm Review: Induction

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2.$$

Prove that for all $n \in \mathbb{N}$, $S_n = \frac{1}{6}n(n+1)(2n+1)$.

4. Induction with Formulas

These problems are a little more difficult and abstract. Try making sure you can do all the other problems before trying these ones.

(a) (i) Show that given two sets A and B that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (Don't use induction.)

(ii) Show using induction that for an integer $n \geq 2$, given n sets $A_1, A_2, \dots, A_{n-1}, A_n$ that

$$\overline{A_1 \cup A_2 \cup \dots \cup A_{n-1} \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_{n-1}} \cap \overline{A_n}$$

(b) (i) Show that given any integers a , b , and c , if $c \mid a$ and $c \mid b$, then $c \mid (a + b)$. (Don't use induction.)

(ii) Show using induction that for any integer $n \geq 2$, given n numbers $a_1, a_2, \dots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \dots, n$, that

$$c \mid (a_1 + a_2 + \dots + a_{n-1} + a_n).$$

In other words, if a number divides each term in a sum then that number divides the sum.

5. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in year n is described by the function $f(n)$:

$$f(0) = 0$$

$$f(1) = 1$$

$$f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n . That is, construct a formula for $f(n)$ and prove its correctness.

6. Walk the Dawgs

Suppose a dog walker takes care of $n \geq 12$ dogs. The dog walker is not a strong person, and will walk dogs in groups of 3 or 7 at a time (every dog gets walked exactly once). Prove the dog walker can always split the n dogs into groups of 3 or 7.