Section 06: Set Theory and Induction

1. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

(b) $B = \{\{\},\ \{\{\}\},\ \{\{\},\{\}\},\ \{\{\},\{\},\{\}\},\ \dots\}$

(c) $D = \varnothing$

(d) $E = \{\varnothing\}$

(e) $C = A \times (B \cup \{7\})$

2. Game, Set, Match

Prove each of the following set identities, formally and then in English.

(a) $A \setminus B \subseteq A \cup C$ for any sets A, B, C.

(b) $(A \setminus B) \setminus C \subseteq A \setminus C$ for any sets A, B.

(c) $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

3. Set Equality

Let A and B be sets. Consider the claim: $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

State what the claim becomes when you unroll the definition of “=“ sets. Then, following the Meta Theorem template, write an English proof that the claim holds.

4. Power Sets

Let A and B be sets. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ follows from $A \subseteq B$.

Write a formal proof. Then, translate it to English one.

5. Ghosts and Skeletons

Let A and B be sets and P and Q be predicates. For each of the claims below, write the *skeleton* of an English proof of the claim. It will not be possible to complete the proof with just the information given, but you should be able to see the basic shape of the proof.

For example, suppose we want to prove "No element of A satisfies P." Then, our proof would have this shape:

> Let x be arbitrary.
> Suppose that $x \in A$. **....** Thus, $P(x)$ is false.
> Since x was arbitrary, this shows that no element of A satisfies P.

This shows the general shape (skeleton) of the proof. We don't know how to complete the proof since we don't know what A and P are. For any particular choice of A and P, though, the proof would still look like this but with the "...." replaced by specific reasoning for that A and P.

Note that we have actually proven $\forall x \, \neg P(x)$, whereas the claim best translates as $\neg \exists x \, P(x)$. However, the two are equivalent by De Morgan's law, and that is a simple enough step that the reader should see it.

(a) $A = B$

(b) Any object that satisfies P but not Q is in the set B.

(c) B is not a subset of A.

6. A Horse of a Different Color

Did you know that all dogs are named Dubs? It's true. Maybe. Let's prove it by induction. The key is talking about groups of dogs, where every dog has the same name.

Let $P(i)$ mean "all groups of i dogs have the same name." We prove $\forall n \, P(n)$ by induction on n.

Base Case: $P(1)$ Take an arbitrary group of one dog, all dogs in that group all have the same name (there's only the one, so it has the same name as itself).

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary k.

Inductive Step: Consider an arbitrary group of $k+1$ dogs. Arbitrarily select a dog, D, and remove it from the group. What remains is a group of k dogs. By inductive hypothesis, all k of those dogs have the same name. Add D back to the group, and remove some other dog D'. We have a (different) group of k dogs, so the inductive hypothesis applies again, and every dog in that group also shares the same name. All $k+1$ dogs appeared in at least one of the two groups, and our groups overlapped, so all of our $k+1$ dogs have the same name, as required.

Conclusion: We conclude $P(n)$ holds for all n by the principle of induction.

Recalling that Dubs is a dog, we have that every dog must have the same name as him, so every dog is named Dubs.

This proof cannot be correct (the proposed claim is false). Where is the bug?

7. Induction with Equality

(a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

(b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

8. Induction with Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

9. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.