Section 06: Solutions

1. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

Solution:

3

(b) $B = \{\{\},\ \{\{\}\},\ \{\{\},\{\}\},\ \{\{\},\{\},\{\}\},\ \dots\}$

Solution:

$$\begin{aligned} B &= \{\{\},\ \{\{\}\},\ \{\{\},\{\}\},\ \{\{\},\{\},\{\}\},\ \dots\} \\ &= \{\{\},\ \{\{\}\},\ \{\{\}\},\ \{\{\}\},\ \dots\} \\ &= \{\varnothing, \{\varnothing\}\} \end{aligned}$$

So, there are two elements in B.

(c) $D = \varnothing$

Solution:

0.

(d) $E = \{\varnothing\}$

Solution:

1.

(e) $C = A \times (B \cup \{7\})$

Solution:

$C = \{1, 2, 3\} \times \{\varnothing, \{\varnothing\}, 7\} = \{(a, b) \mid a \in \{1, 2, 3\}, b \in \{\varnothing, \{\varnothing\}, 7\}\}$. It follows that there are $3 \times 3 = 9$ elements in C.

2. Game, Set, Match

Prove each of the following set identities, formally and then in English.

(a) $A \setminus B \subseteq A \cup C$ for any sets A, B, C.

Solution:

1.	Let x be an arbitrary object.	
2.1.	$x \in A \setminus B$	Assumption
2.2.	$(x \in A) \land \neg(x \in B)$	Def of "$\setminus$": 2.1
2.3.	$x \in A$	Elim $\land$: 2.2
2.4.	$(x \in A) \lor (x \in C)$	Intro $\lor$: 2.3
2.5.	$x \in A \cup C$	Def of $\cup$: 2.4
2.	$(x \in A \setminus B) \to (x \in A \cup C)$	Direct Proof
3.	$\forall x\,((x \in A \setminus B) \to (x \in A \cup C))$	Intro $\forall$: 1, 2
4.	$A \setminus B \subseteq A \cup C$	Def of $\subseteq$: 3

Let x be an arbitrary object. Suppose that $x \in A \setminus B$. By definition, this means that $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in A \cup C$ by the definition of $\cup$. Since x was arbitrary, this shows $A \setminus B \subseteq A \cup C$.

(b) $(A \setminus B) \setminus C \subseteq A \setminus C$ for any sets A, B.

Solution:

1.	Let x be arbitrary.	
2.1.	$x \in (A \setminus B) \setminus C$	Assumption
2.2.	$(x \in A \setminus B) \land (x \notin C)$	Def of "$\setminus$": 2.1
2.3.	$(x \in A) \land (x \notin B)$	Def of "$\setminus$": 2.2
2.4.	$x \in A$	Elim $\land$: 2.3
2.5.	$x \notin C$	Elim $\land$: 2.2
2.6.	$(x \in A) \land (x \notin C)$	Intro $\land$: 2.4, 2.5
2.7.	$x \in A \setminus C$	Def of "$\setminus$": 2.6
2.	$(x \in (A \setminus B) \setminus C) \to (x \in A \setminus C)$	Direct Proof
3.	$\forall x\,((x \in (A \setminus B) \setminus C) \to (x \in A \setminus C)$	Intro $\forall$: 1, 2
4.	$(A \setminus B) \setminus C \subseteq A \setminus C$	Def of $\subseteq$: 3

Let x be an arbitrary object. Suppose that $x \in (A \setminus B) \setminus C$. By definition, this means that $x \in A \setminus B$ and $x \notin C$ and then that $x \in A$ and $x \notin B$. The facts that $x \in A$ and $x \notin C$ show that $x \in A \setminus C$ by definition. Since x was arbitrary, this shows $(A \setminus B) \setminus C \subseteq A \setminus C$.

(c) $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Solution:

1.	Let x be an arbitrary object.	
2.1.	$x \in (A \cap B) \times C$	Assumption
2.2.	$\exists y \exists z((y \in (A \cap B)) \wedge (z \in C) \wedge (x = (y, z)))$	Def of Cartesian Product: 2.1
2.3.	$\exists z((y \in (A \cap B)) \wedge (z \in C) \wedge (x = (y, z)))$	Elim $\exists$: 2.2
2.4.	$((y \in (A \cap B)) \wedge (z \in C) \wedge (x = (y, z)))$	Elim $\exists$: 2.3
2.5.	$y \in (A \cap B)$	Elim $\wedge$: 2.4
2.6.	$y \in A \wedge y \in B$	Def of Intersection: 2.5
2.7.	$y \in A$	Elim $\wedge$: 2.6
2.8.	$z \in C$	Elim $\wedge$: 2.4
2.9.	$z \in C \vee z \in D$	Intro $\vee$: 2.8
2.10.	$z \in (C \cup D)$	Def of Union: 2.9
2.11.	$x = (y, z)$	Elim $\wedge$: 2.4
2.12.	$(y \in A) \wedge (z \in (C \cup D)) \wedge (x = (y, z))$	Intro $\wedge$: 2.7, 2.10, 2.11
2.13.	$\exists z((y \in A) \wedge (z \in (C \cup D)) \wedge (x = (y, z)))$	Intro $\exists$: 2.12
2.14.	$\exists y \exists z((y \in A) \wedge (z \in (C \cup D)) \wedge (x = (y, z)))$	Intro $\exists$: 2.13
2.15.	$x \in A \times (C \cup D)$	Def of Cartesian Product: 2.14
2.	$(x \in (A \cap B) \times C) \rightarrow (x \in A \times (C \cup D))$	Direct Proof
3.	$\forall x\, (x \in (A \cap B) \times C) \rightarrow (x \in A \times (C \cup D))$	Intro $\forall$: 1, 2
4.	$(A \cap B) \times C \subseteq A \times (C \cup D)$	Def of Subset

Let x be an arbitrary element of $(A \cap B) \times C$. Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$. Since $y \in A \cap B$, by definition of $\cap$, $y \in A$ (and $y \in B$). Since $z \in C$, by definition of $\cup$, we also have $z \in C \cup D$. Thus, since $y \in A$ and $z \in C \cup D$, by definition of Cartesian product we have $x = (y, z) \in A \times (C \cup D)$. Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

3. Set Equality

Let A and B be sets. Consider the claim: $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

State what the claim becomes when you unroll the definition of "=" sets. Then, following the Meta Theorem template, write an English proof that the claim holds.

Solution:

Unrolling the $=$, the claim is: $\forall x((x \in A \setminus (B \cup C)) \leftrightarrow (x \in (A \setminus B) \cap (A \setminus C)))$.

Let x be arbitrary.

$$
\begin{aligned}
x \in A \setminus (B \cup C) &\equiv x \in A \wedge \neg(x \in (B \cup C)) && \text{[Def of Set Difference]}\\
&\equiv x \in A \wedge \neg(x \in B \vee x \in C) && \text{[Def of Union]}\\
&\equiv x \in A \wedge (x \notin B \wedge x \notin C) && \text{[De Morgan]}\\
&\equiv (x \in A \wedge x \in A) \wedge (x \notin B \wedge x \notin C) && \text{[Idempotency]}\\
&\equiv (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) && \text{[Associativity/Commutativity]}\\
&\equiv (x \in (A \setminus B)) \wedge (x \in (A \setminus C)) && \text{[Def of Set Difference]}\\
&\equiv (x \in (A \setminus B) \cap (A \setminus C)) && \text{[Def of Intersection]}
\end{aligned}
$$

Since x was arbitrary, we have shown that the two sets contain the same elements.

4. Power Sets

Let A and B be sets. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ follows from $A \subseteq B$.

Write a formal proof. Then, translate it to English one.

Solution:

We can prove that as follows:

1. $A \subseteq B$ — Given
2. $\forall x\,((x \in A) \to (x \in B))$ — Def of Subset: 1
   - 3.1. Let X be arbitrary
     - 3.2.1 $X \in \mathcal{P}(A)$ — Assumption
     - 3.2.2 $X \subseteq A$ — Def of Power Set: 3.2.1
     - 3.2.3 $\forall x\,((x \in X) \to (x \in A))$ — Def of Subset: 3.2.2
     - 3.2.4 Let x be arbitrary
       - 3.2.5.1 $x \in X$ — Assumption
       - 3.2.5.2 $(x \in X) \to (x \in A)$ — Elim $\forall$: 3.2.3
       - 3.2.5.3 $(x \in A)$ — Modus Ponens: 3.2.5.1, 3.2.5.2
       - 3.2.5.4 $(x \in A) \to (x \in B)$ — Elim $\forall$: 2
       - 3.2.5.5 $x \in B$ — Modus Ponens: 3.2.5.3, 3.2.5.4
     - 3.2.5 $\forall x\,((x \in X) \to (x \in B))$ — Intro $\forall$
     - 3.2.6 $X \subseteq B$ — Undef Subset: 3.2.5
     - 3.2.7 $X \in \mathcal{P}(B)$ — Undef Power Set: 3.2.6
   - 3.2. $(X \in \mathcal{P}(A)) \to (X \in \mathcal{P}(B))$ — Direct Proof
3. $\forall X\,((X \in \mathcal{P}(A)) \to (X \in \mathcal{P}(B)))$ — Intro $\forall$
4. $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ — Def of Subset: 3

Let X be an arbitrary set in $\mathcal{P}(A)$. By definition of power set, $X \subseteq A$. We need to show that $X \in \mathcal{P}(B)$, or equivalently, that $X \subseteq B$.

Let x be an arbitrary element of X. Since $X \subseteq A$, it must be the case that $x \in A$. We were given that $A \subseteq B$. By definition of subset, any element of A is an element of B. So, it must also be the case that $x \in B$.

Since x was arbitrary, we know any element of X is an element of B. By definition of subset, $X \subseteq B$. By definition of power set, $X \in \mathcal{P}(B)$.

Since X was an arbitrary set, any set in $\mathcal{P}(A)$ is in $\mathcal{P}(B)$, or, by definition of subset, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. We have shown the claim.

5. Ghosts and Skeletons

Let A and B be sets and P and Q be predicates. For each of the claims below, write the *skeleton* of an English proof of the claim. It will not be possible to complete the proof with just the information given, but you should be able to see the basic shape of the proof.

For example, suppose we want to prove "No element of A satisfies P." Then, our proof would have this shape:

Let x be arbitrary.
Suppose that $x \in A$. **....** Thus, $P(x)$ is false.
Since x was arbitrary, this shows that no element of A satisfies P.

This shows the general shape (skeleton) of the proof. We don't know how to complete the proof since we don't know what A and P are. For any particular choice of A and P, though, the proof would still look like this but with the "...." replaced by specific reasoning for that A and P.

Note that we have actually proven $\forall x \, \neg P(x)$, whereas the claim best translates as $\neg \exists x \, P(x)$. However, the two are equivalent by De Morgan's law, and that is a simple enough step that the reader should see it.

(a) $A = B$

Solution:

Let x be arbitrary.

Suppose that $x \in A$. **....** Thus, $x \in B$.

Now, suppose that $x \in B$. **....** Thus, $x \in A$.

We have shown that $x \in A$ iff $x \in B$. Since x was arbitrary, the sets are equal by definition.

(b) Any object that satisfies P but not Q is in the set B.

Solution:

Let x be arbitrary.

Suppose that x satisfies P but not Q. **....** Thus, $x \in B$.

Since x was arbitrary, this shows that anything satisfying P but not Q is in B.

(c) B is not a subset of A.

Solution:

[We need to show $\neg \forall x \, ((x \in B) \to (x \in A))$, but that is equivalent to $\exists x \, ((x \in B) \wedge (x \notin A))$.]

Let $x = \ldots$. Since ..., we can see that $x \in B$. On the other hand, since ..., we can see that $x \notin A$. Thus, x is a counterexample to the claim that B is a subset of A.

6. A Horse of a Different Color

Did you know that all dogs are named Dubs? It's true. Maybe. Let's prove it by induction. The key is talking about groups of dogs, where every dog has the same name.

Let $P(i)$ mean "all groups of i dogs have the same name." We prove $\forall n \, P(n)$ by induction on n.

Base Case: $P(1)$ Take an arbitrary group of one dog, all dogs in that group all have the same name (there's only the one, so it has the same name as itself).

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary k.

Inductive Step: Consider an arbitrary group of $k+1$ dogs. Arbitrarily select a dog, D, and remove it from the group. What remains is a group of k dogs. By inductive hypothesis, all k of those dogs have the same name. Add D back to the group, and remove some other dog D'. We have a (different) group of k dogs, so the inductive hypothesis applies again, and every dog in that group also shares the same name. All $k+1$ dogs appeared in at least one of the two groups, and our groups overlapped, so all of our $k+1$ dogs have the same name, as required.

Conclusion: We conclude $P(n)$ holds for all n by the principle of induction.

Recalling that Dubs is a dog, we have that every dog must have the same name as him, so every dog is named Dubs.

This proof cannot be correct (the proposed claim is false). Where is the bug?

Solution:

The bug is in the final sentence of the inductive step. We claimed that the groups overlapped, i.e. that some dog was in both of them. That's true for large k, but not when $k+1=2$. When $k=2$, D is in a group by itself, and D' was in a group by itself. The inductive hypothesis holds (D has the only name in its subgroup, and D' has the only name in its subgroup) but returning to the full group $\{D, D'\}$ we cannot conclude that they share a name.

From there everything unravels. $P(1) \not\to P(2)$, so we cannot use the principle of induction. It turns out this is the **only** bug in the proof. The argument in the inductive step is correct as long as $k+1>2$. But that implication is always vacuous, since $P(2)$ is false.

7. Induction with Equality

(a) Show using induction that $0+1+2+\cdots+n=\frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Solution:

For $n \in \mathbb{N}$ let $P(n)$ be "$0+1+\cdots+n=\frac{n(n+1)}{2}$". We show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case: We have $0=\frac{0(0+1)}{2}$ which is $P(0)$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary integer $k \geq 0$.

Inductive Step: $\boxed{\text{Goal: Show } 0+1+\cdots+(k+1)=\dfrac{(k+1)(k+2)}{2}}$.

We have

$$\begin{aligned}
0+1+\cdots+k+(k+1) &= (0+1+\cdots+k)+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) && \text{[Inductive Hypothesis]} \\
&= \frac{k(k+1)}{2}+\frac{2(k+1)}{2} \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} && \text{[Factor out } (k+1)\text{]}
\end{aligned}$$

This proves $P(k+1)$.

Conclusion: $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

(b) Define the triangle numbers as $\triangle_n=1+2+\cdots+n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n=\frac{n(n+1)}{2}$.

Prove the following equality for all $n \in \mathbb{N}$:

$$0^3+1^3+\cdots+n^3=\triangle_n^2$$

Solution:

First, note that $\triangle_n=(0+1+2+\cdots+n)$. So, we are trying to prove $(0^3+1^3+\cdots+n^3)=(0+1+\cdots+n)^2$. Let $P(n)$ be the statement:

$$0^3+1^3+\cdots+n^3=(0+1+\cdots+n)^2.$$

We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case. $0^3 = 0^2$, so $P(0)$ holds.

Inductive Hypothesis. Suppose that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$.

Inductive Step. We show $P(k+1)$:

$$\begin{aligned}
0^3 + 1^3 + \cdots (k+1)^3 &= (0^3 + 1^3 + \cdots + k^3) + (k+1)^3 && \text{[Associativity]} \\
&= (0 + 1 + \cdots + k)^2 + (k+1)^3 && \text{[Inductive Hypothesis]} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[Part (a)]} \\
&= (k+1)^2 \left(\frac{k^2}{2^2} + (k+1)\right) && \text{[Factor } (k+1)^2\text{]} \\
&= (k+1)^2 \left(\frac{k^2 + 4k + 4}{4}\right) && \text{[Add via common denominator]} \\
&= (k+1)^2 \left(\frac{(k+2)^2}{4}\right) && \text{[Factor numerator]} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 && \text{[Take out the square]} \\
&= (0 + 1 + \cdots + (k+1))^2 && \text{[Part (a)]}
\end{aligned}$$

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of induction.

8. Induction with Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction. **Solution:**

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case ($n = 2$)**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Induction Hypothesis: Assume that $9 \mid j^3 + (j+1)^3 + (j+2)^3$ for an arbitrary integer $j > 1$. Note that this is equivalent to assuming that $j^3 + (j+1)^3 + (j+2)^3 = 9k$ for some integer k by the definition of divides.

Induction Step: $\boxed{\text{Goal: Show } 9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3}$

$$\begin{aligned}
(j+1)^3 + (j+2)^3 + (j+3)^3 &= (j+3)^3 + 9k - j^3 \text{ for some integer } k && \text{[Induction Hypothesis]} \\
&= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\
&= 9j^2 + 27j + 27 + 9k \\
&= 9(j^2 + 3j + 3 + k)
\end{aligned}$$

Since j is an integer, $j^2 + 3j + 3 + k$ is also an integer. Therefore, by the definition of divides, $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$, so $P(j) \to P(j+1)$ for an arbitrary integer $j > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

9. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$. **Solution:**

Let $P(n)$ be "$6n + 6 < 2^n$". We will prove $P(n)$ for all integers $n \geq 6$ by induction on n

Base Case $(n = 6)$**:** $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

Inductive Hypothesis: Assume that $6k + 6 < 2^k$ for an arbitrary integer $k \geq 6$.

Inductive Step: $\boxed{\text{Goal: Show } 6(k+1) + 6 < 2^{k+1}}$

$$
\begin{aligned}
6(k+1) + 6 &= 6k + 6 + 6 \\
&< 2^k + 6 && \text{[Inductive Hypothesis]} \\
&< 2^k + 2^k && \text{[Since } 2^k > 6 \text{, since } k \geq 6] \\
&= 2 \cdot 2^k \\
&= 2^{k+1}
\end{aligned}
$$

So $P(k) \to P(k+1)$ for an arbitrary integer $k \geq 6$.

Conclusion: $P(n)$ holds for all integers $n \geq 6$ by the principle of induction.