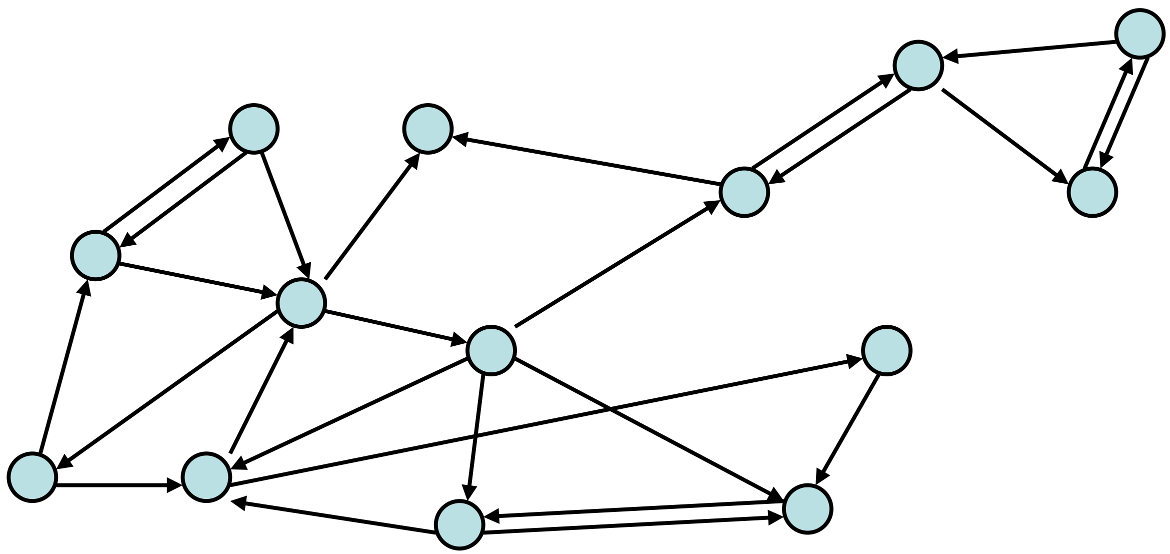
CSE 311: Foundations of Computing

Lecture 22: Relations and Directed Graphs



Last time: Languages — REs and CFGs

Saw two new ways of defining languages

- Regular Expressions $(0 \cup 1)^*\ 0110\ (0 \cup 1)^*$
  - easy to understand (declarative)
- Context-free Grammars $S \rightarrow SS \mid 0S1 \mid 1S0 \mid \varepsilon$
  - more expressive
  - (≈ recursively-defined sets)

We will connect these to machines shortly.

But first, we need some new math terminology….

Alternative Set Notation

We defined Cartesian Product as

$$\boxed{A \times B ::= \{x : \exists a \in A, \exists b \in B\ (x = (a, b))\ \}}$$

Alternative notation for this is

$$\boxed{A \times B ::= \{(a, b) : a \in A, b \in B\}}$$

“The set of all (a, b) such that a ∈ A and b ∈ B”

Relations

Let A and B be sets,
A **binary relation from** A **to** B is a subset of $A \times B$

Let A be a set,
A **binary relation on** A is a subset of $A \times A$

Relations You Already Know

$\geq$ on $\mathbb{N}$

That is: $\{(x,y) : x \geq y \text{ and } x, y \in \mathbb{N}\}$

$<$ on $\mathbb{R}$

That is: $\{(x,y) : x < y \text{ and } x, y \in \mathbb{R}\}$

$=$ on Σ^*

That is: $\{(x,y) : x = y \text{ and } x, y \in \Sigma^*\}$

$\subseteq$ on $\mathcal{P}(U)$ for universe U

That is: $\{(A,B) : A \subseteq B \text{ and } A, B \in \mathcal{P}(U)\}$

More Relation Examples

$\mathbf{R_1}$ **=** {(a, 1), (a, 2), (b, 1), (b, 3), (c, 3)}

$\mathbf{R_2}$ **=** {(x, y) : $x \equiv_5 y$ }

$\mathbf{R_3}$ **=** {(c_1, c_2) : c_1 is a prerequisite of c_2 }

$\mathbf{R_4}$ **=** {(s, c) : student s has taken course c }

Properties of Relations

Let R be a relation on A.

R is **reflexive** iff $(a,a) \in R$ for every $a \in A$

R is **symmetric** iff $(a,b) \in R$ implies $(b,a) \in R$

R is **antisymmetric** iff $(a,b) \in R$ and $a \neq b$ implies $(b,a) \notin R$

R is **transitive** iff $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$

Which relations have which properties?

$\geq$ **on** $\mathbb{N}$ **:**

$<$ **on** $\mathbb{R}$ **:**

$=$ **on** Σ^* **:**

$\subseteq$ **on** $\mathcal{P}$**(U):**

R_2 = $\{(x, y) : x \equiv_5 y\}$:

R_3 = $\{(c_1, c_2) : c_1$ is a prerequisite of $c_2\}$:

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Which relations have which properties?

$\geq$ on $\mathbb{N}$: Reflexive, Antisymmetric, Transitive

$<$ on $\mathbb{R}$: Antisymmetric, Transitive

$=$ on Σ^* : Reflexive, Symmetric, Antisymmetric, Transitive

$\subseteq$ on $\mathcal{P}(U)$: Reflexive, Antisymmetric, Transitive

$R_2 = \{(x, y) : x \equiv_5 y\}$: Reflexive, Symmetric, Transitive

$R_3 = \{(c_1, c_2) : c_1$ is a prerequisite of $c_2\}$: Antisymmetric

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Combining Relations

Let R be a relation from A to B.
Let S be a relation from B to C.

The composition of R and S, $R \circ S$ is the relation from A to C defined by:

$$R \circ S = \{(a, c) : \exists \text{ b such that } (a, b) \in R \text{ and } (b, c) \in S\}$$

Intuitively, a pair is in the composition if there is a "connection" from the first to the second.

Examples

(a,b) $\in$ Parent iff b is a parent of a

(a,b) $\in$ Sister iff b is a sister of a

When is (x,y) $\in$ Parent $\circ$ Sister?

When is (x,y) $\in$ Sister $\circ$ Parent?

$R \circ S = \{(a, c) : \exists$ b such that $(a,b) \in R$ and $(b,c) \in S\}$

Examples

Using only the relations Parent, Child, Father, Son, Brother, Sibling, Husband

and *composition*, express the following:

Uncle: b is an uncle of a

Cousin: b is a cousin of a

Powers of a Relation

$$\boldsymbol{R^2} ::= \boldsymbol{R} \circ \boldsymbol{R}$$
$$= \{(\boldsymbol{a}, \boldsymbol{c}) : \exists \boldsymbol{b} \text{ such that } (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R} \text{ and } (\boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{R}\}$$

$\boldsymbol{R^0} ::= \{(\boldsymbol{a}, \boldsymbol{a}) : \boldsymbol{a} \in \boldsymbol{A}\}$ "the equality relation on $\boldsymbol{A}$"

$\boldsymbol{R^{n+1}} ::= \boldsymbol{R^n} \circ \boldsymbol{R}$ for $\boldsymbol{n} \geq \boldsymbol{0}$

e.g., $\boldsymbol{R^1} = \boldsymbol{R^0} \circ \boldsymbol{R} = \boldsymbol{R}$

$\boldsymbol{R^2} = \boldsymbol{R^1} \circ \boldsymbol{R} = \boldsymbol{R} \circ \boldsymbol{R}$

Non-constructive Definitions

Recursively defined sets and functions describe these objects by explaining how to construct / compute them

But sets can also be defined non-constructively:

$$\boxed{S = \{x : P(x)\}}$$

How can we define functions non-constructively?

- (useful for writing a function specification)

Functions

A function $f : A \to B$ (A as input and B as output) is a special type of relation.

A **function** f **from** A **to** B is a relation from A to B such that: for every $a \in A$, there is *exactly one* $b \in B$ with $(a, b) \in f$

I.e., for every input $a \in A$, there is one output $b \in B$.
We denote this b by $f(a)$.

(When attempting to define a function this way, we sometimes say the function is "well defined" if the *exactly one* part holds)

Functions

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Ex: {((a, b), d) : d is the largest integer dividing a and b}

- **gcd :** $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$
- **defined without knowing how to compute it**

Matrix Representation

Relation $\boldsymbol{R}$ on $\boldsymbol{A} = \{a_1, \ldots, a_p\}$

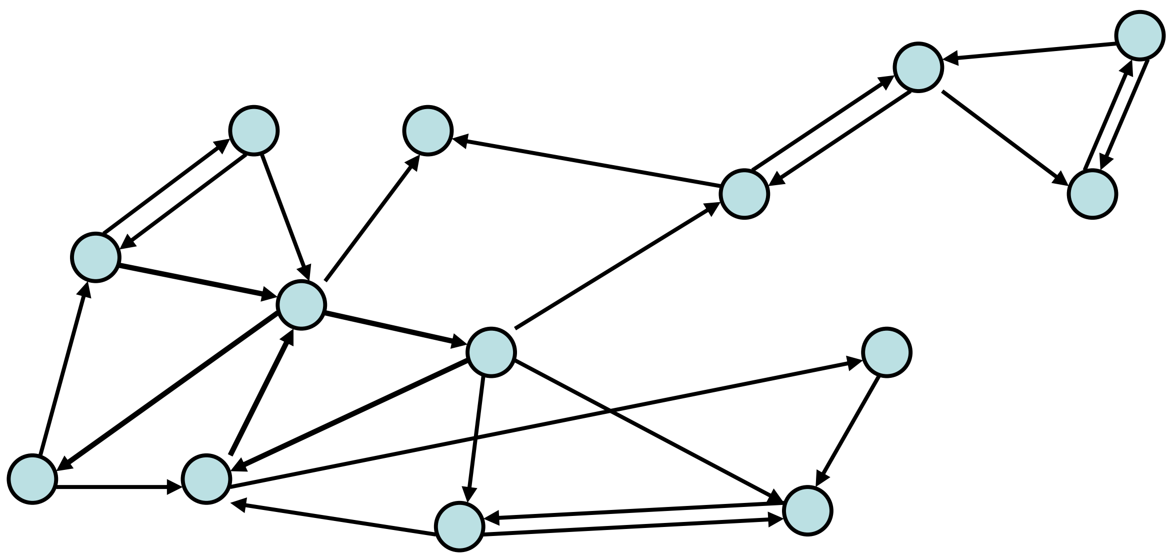
$$\boldsymbol{m_{ij}} = \begin{cases} 1 & \text{if } \left(a_i, a_j\right) \in \boldsymbol{R} \\ 0 & \text{if } \left(a_i, a_j\right) \notin \boldsymbol{R} \end{cases}$$

{ (1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3) }

	1	2	3	4
1	1	1	0	1
2	1	0	1	0
3	0	1	1	0
4	0	1	1	0

Directed Graphs

$G = (V, E)$ V – vertices
E – edges (relation on vertices)



Directed Graphs

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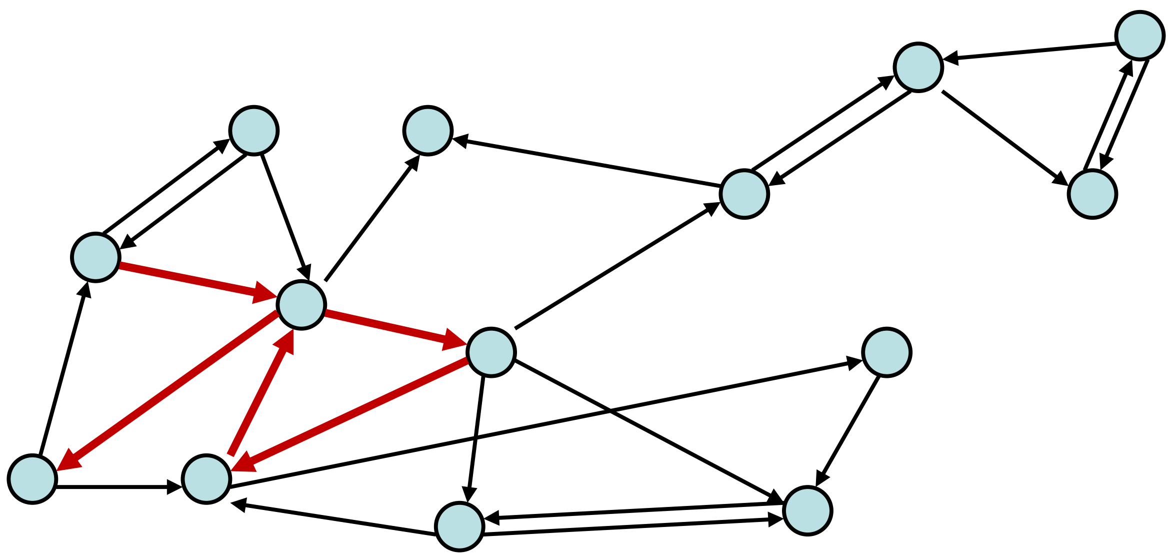
Path: $v_0, v_1, \ldots, v_k$ with each (v_i, v_{i+1}) in E



Directed Graphs

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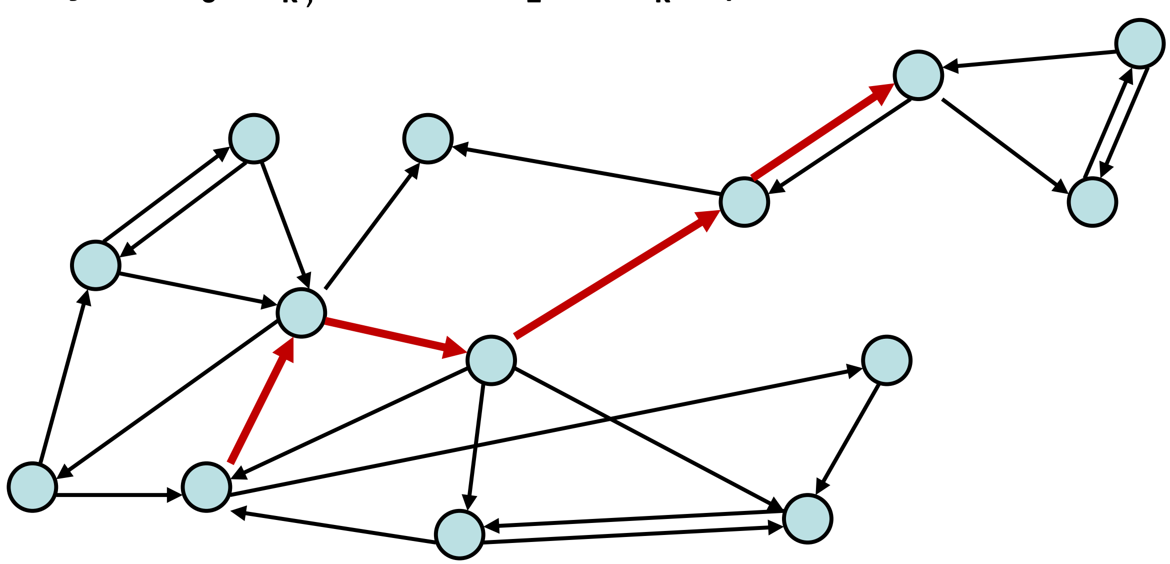
Path: $v_0, v_1, \ldots, v_k$ with each (v_i, v_{i+1}) in E

Simple Path: none of $\mathbf{v_0}, \ldots, \mathbf{v_k}$ repeated
Cycle: $\mathbf{v_0} = \mathbf{v_k}$
Simple Cycle: $\mathbf{v_0} = \mathbf{v_k}$, none of $\mathbf{v_1}, \ldots, \mathbf{v_k}$ repeated



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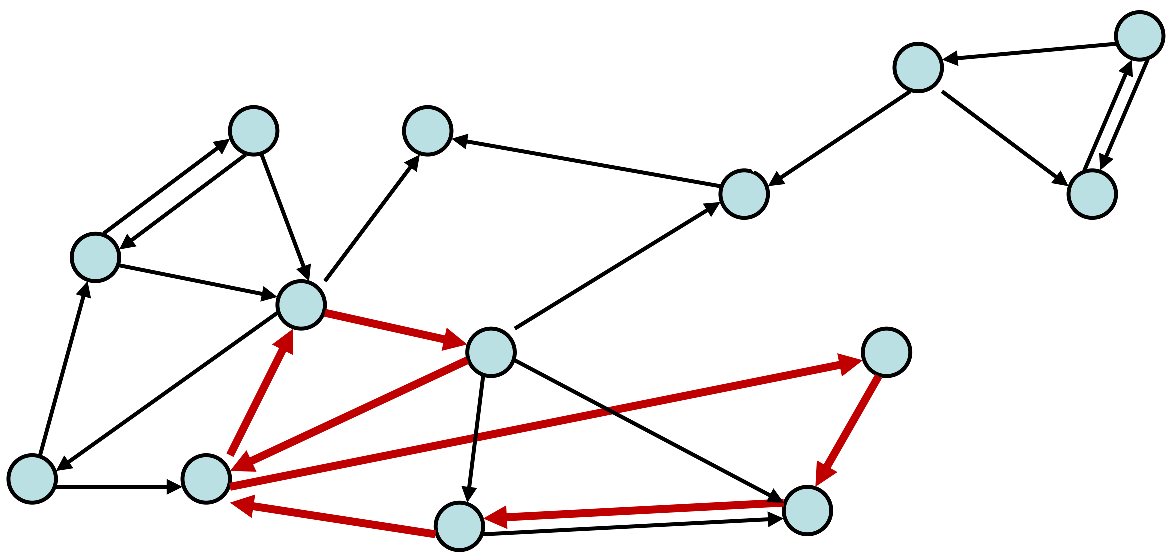

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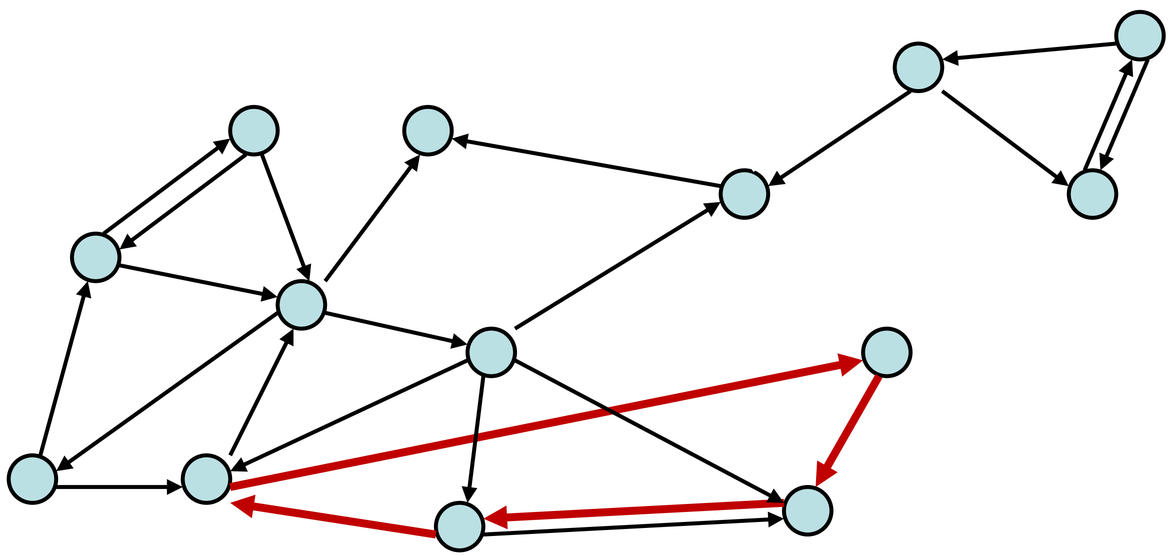
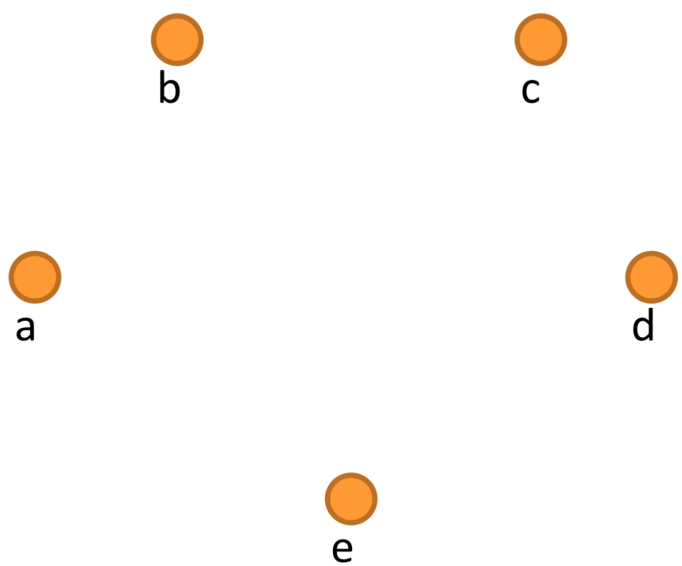


Representation of Relations

Directed Graph Representation (Digraph)

{(a, b), (a, a), (b, a), (c, a), (c, d), (c, e) (d, e) }



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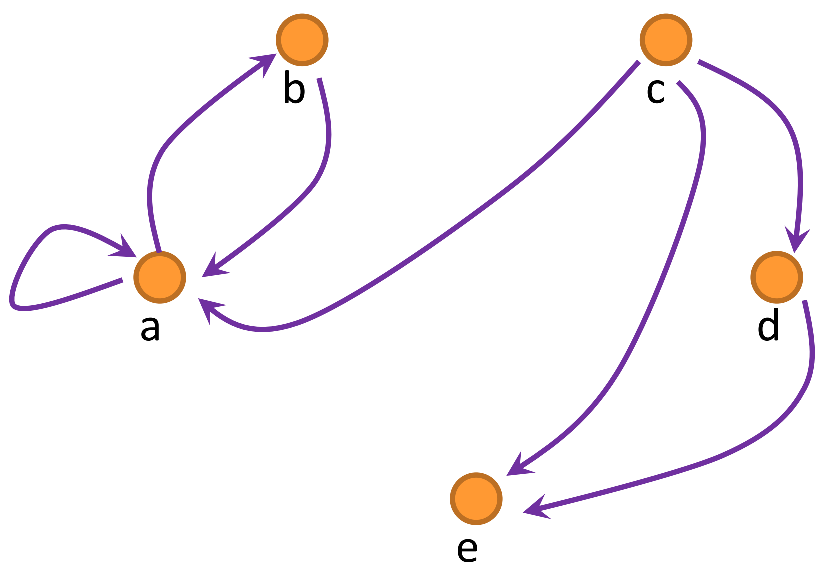
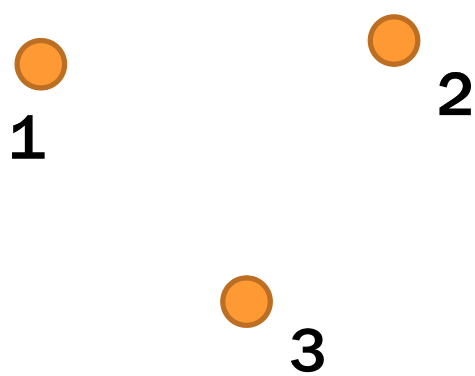
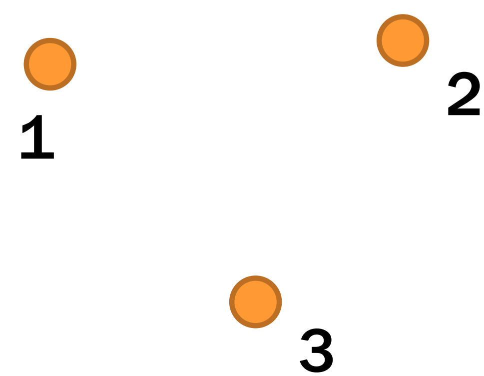


Relational Composition using Digraphs

If $\boldsymbol{S} = \{(2,2),(2,3),(3,1)\}$ and $\boldsymbol{R} = \{(1,2),(2,1),(1,3)\}$

Compute $\boldsymbol{R} \circ \boldsymbol{S}$





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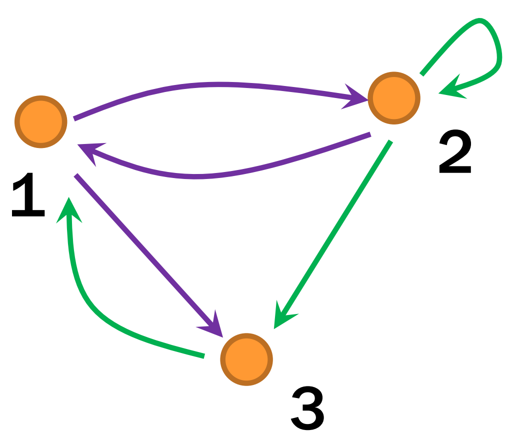
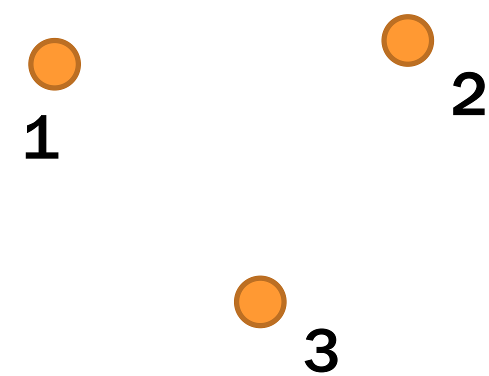
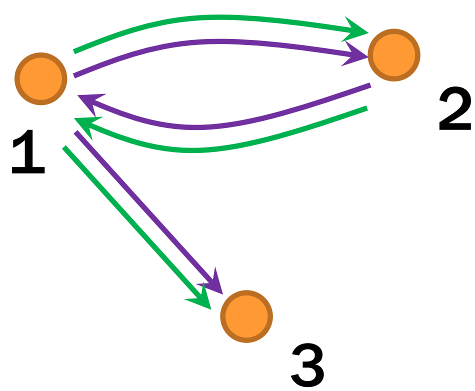


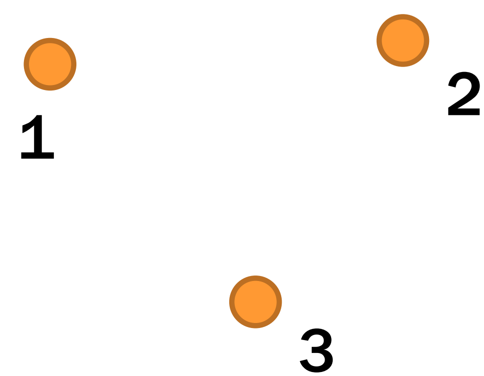


Relational Composition using Digraphs

If $\boldsymbol{R = \{(1,2),(2,1),(1,3)\}}$ and $\boldsymbol{R = \{(1,2),(2,1),(1,3)\}}$

Compute $\boldsymbol{R \circ R}$





$(a,c) \in R \circ R = R^2$ iff $\exists b\ ((a,b) \in R \land (b,c) \in R)$

iff $\exists b$ such that a, b, c is a path

Relational Composition using Digraphs

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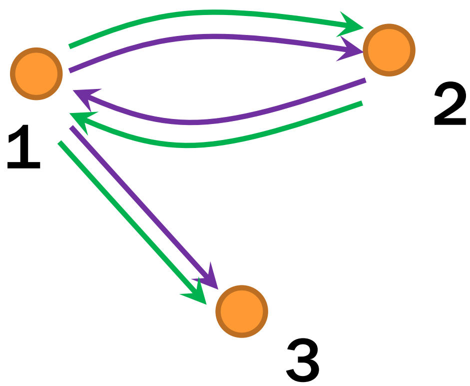

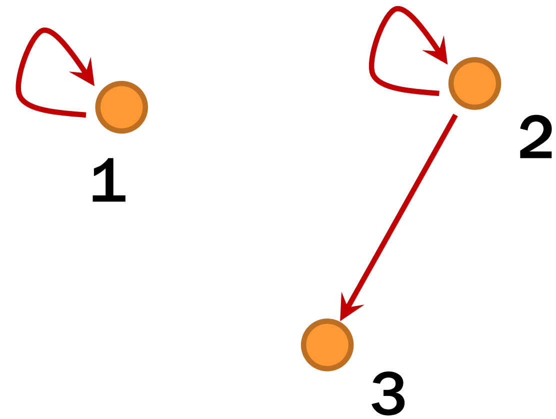

$(a,c) \in R \circ R = R^2$ iff $\exists b \; ((a,b) \in R \wedge (b,c) \in R)$

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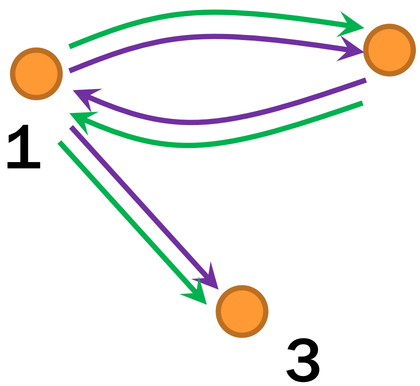

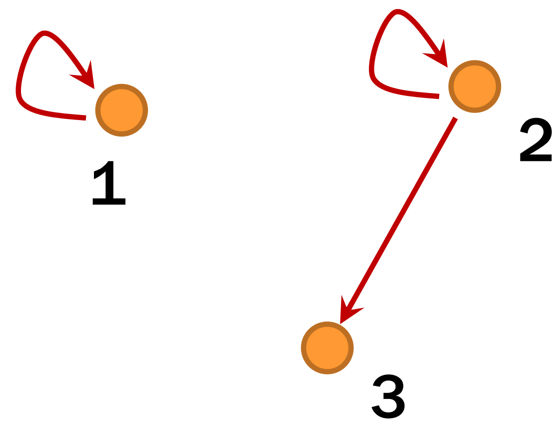

Special case: $R \circ R$ is paths of length 2.

- R is paths of length 1
- R^0 is paths of length 0 (can't go anywhere)
- $R^3 = R^2 \circ R$ etc, so is R^n paths of length n