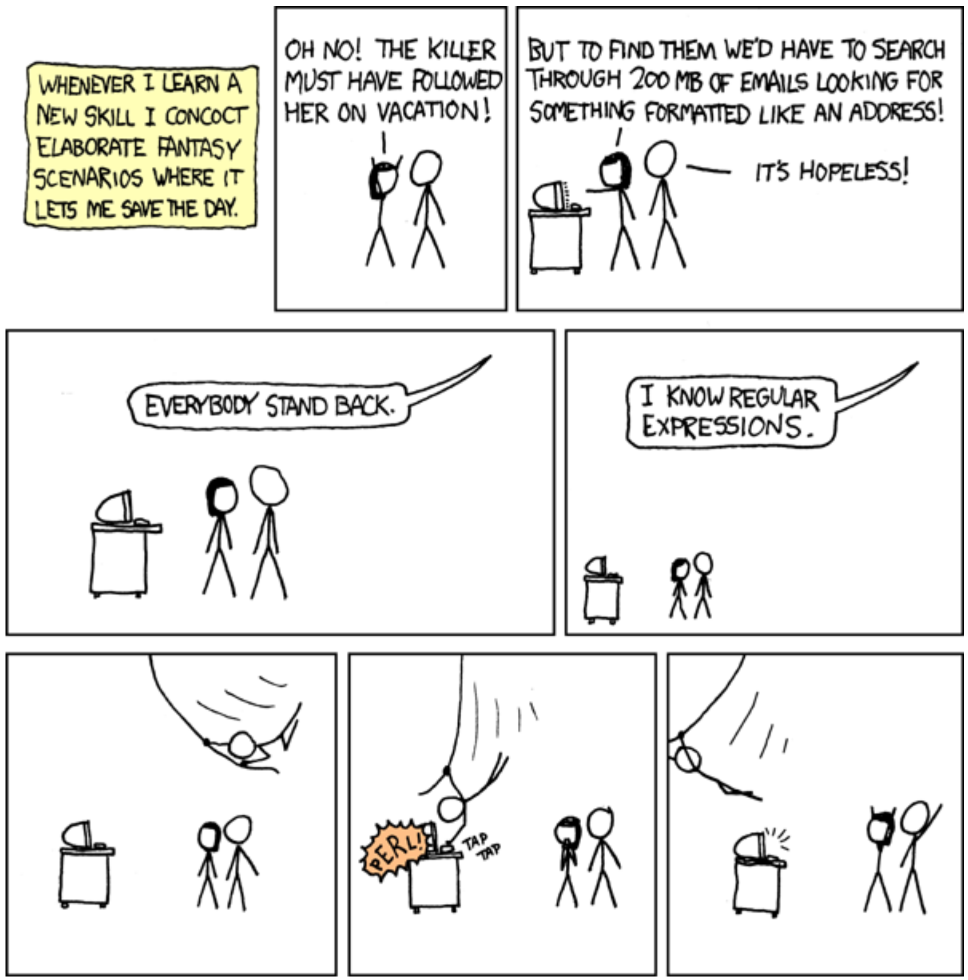
CSE 311: Foundations of Computing

Lecture 20: Structural Induction, Regular Expressions



Previously on 311: Recursive Definitions

Examples we saw fall in two categories

- new types of data
- subsets of previously-defined data

New Types	Subsets
Natural Numbers Lists Trees Strings	Even Numbers Powers of 3 Fibonacci Numbers

Last time: Structural Induction

Structural induction is the tool used to prove many more interesting theorems

- General associativity follows from our one rule
  - likewise for generalized De Morgan's laws
- Okay to substitute y for x everywhere in a modular equation when we know that $x \equiv_m y$
- More coming shortly…

Theoretical Computer Science

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ^* of *strings* over the alphabet Σ
  - example: {0,1}* is the set of *binary strings*

    0, 1, 00, 01, 10, 11, 000, 001, … and “”

- Σ^* is defined recursively by
  - Basis: $\varepsilon \in \Sigma^*$ (ε is the empty string, i.e., “”)
  - Recursive: if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Languages: Sets of Strings

- Subsets of strings are called *languages*
- Examples:
  - $\Sigma^* =$ All strings over alphabet Σ
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Binary strings with an equal # of 0's and 1's
  - Legal variable names in Java/C/C++
  - Syntactically correct Java/C/C++ programs
  - Valid English sentences

Foreword on Intro to Theory C.S.

- Look at different ways of defining languages
- See which are more expressive than others
  - i.e., which can define more languages

- Later: connect ways of defining languages to different types of (restricted) computers
  - computers capable of recognizing those languages i.e., distinguishing strings in the language from not

- Consequence: computers that recognize more expressive languages are more powerful

Palindromes

Palindromes are strings that are the same when read backwards and forwards

Basis:

ε is a palindrome

any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome,

then apa is a palindrome for every $a \in \Sigma$

Regular Expressions

Regular expressions over Σ

- **Basis**:

  ε is a regular expression (could also include $\varnothing$)

  a is a regular expression for any $a \in \Sigma$

- **Recursive step**:

  If **A** and **B** are regular expressions, then so are:

  A $\cup$ **B**

  AB

  A*

Each Regular Expression is a "pattern"

ε matches only the **empty string**

a matches only the one-character string *a*

A ∪ **B** matches all strings that either **A** matches or **B** matches (or both)

AB matches all strings that have a first part that **A** matches followed by a second part that **B** matches

A* matches all strings that have any number of strings (even 0) that **A** matches, one after another (**ε** ∪ **A** ∪ **AA** ∪ **AAA** ∪ **...**)

Definition of the *language* matched by a regular expression

Language of a Regular Expression

The language defined by a regular expression:

$$\mathrm{L}(\varepsilon) = \{\varepsilon\}$$

$$\mathrm{L}(a) = \{a\}$$

$$\mathrm{L}(A \cup B) = L(A) \cup L(B)$$

$$\mathrm{L}(AB) = \{x \ : \ \exists y \in L(A), \exists z \in L(B)\ (x = y \bullet z)\}$$

$$\mathrm{L}(A^*) = \bigcup_{n=0}^{\infty} L(A^n)$$

A^n defined recursively by

$$A^0 = \emptyset$$

$$A^{n+1} = A^n A$$

Examples

001*

0*1*

Examples

001*

{00, 001, 0011, 00111, ...}

0*1*

Any number of 0’s followed by any number of 1’s

Examples

$(0 \cup 1)\ 0\ (0 \cup 1)\ 0$

$(0^*1^*)^*$

Examples

$(\mathbf{0} \cup \mathbf{1})\ \mathbf{0}\ (\mathbf{0} \cup \mathbf{1})\ \mathbf{0}$

{0000, 0010, 1000, 1010}

$(\mathbf{0}^*\mathbf{1}^*)^*$

All binary strings

Examples

- All binary strings that contain 0110

$$(\mathbf{0} \cup \mathbf{1})^* \, \mathbf{0110} \, (\mathbf{0} \cup \mathbf{1})^*$$

Examples

- All binary strings that contain 0110

$$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$$

- All binary strings that begin with a string of doubled characters (00 or 11) followed by 01010 or 10001

$$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$$

Examples

- All binary strings that have an even # of 1's

Examples

- All binary strings that have an even # of 1's

  e.g., $0^*(10^*10^*)^*$

Examples

- All binary strings that have an even # of 1's

  e.g., $0^*(10^*10^*)^*$

- All binary strings that *don't* contain 101

Examples

- **All binary strings that have an even # of 1's**

  e.g., $0^*(10^*10^*)^*$

- **All binary strings that *don't* contain 101**

  e.g., $0^*(1 \cup 1000^*)^*(0^* \cup 10^*)$

  at least two 0s between 1s

Regular Expressions in Practice

- Used to define the “tokens”: e.g., legal variable names, keywords in programming languages and compilers
- Used in **`grep`**, a program that does pattern matching searches in UNIX/LINUX
- Pattern matching using regular expressions is an essential feature of PHP
- We can use regular expressions in programs to process strings!

Regular Expressions in Java

- Pattern p = Pattern.compile("a*b");
- Matcher m = p.matcher("aaaaab");
- boolean b = m.matches();

`[01]`	a 0 or a 1	**`^`** start of string	**`$`** end of string				
`[0-9]`	any single digit	**`\.`**	period	**`\,`**	comma	**`\-`**	minus
`.`	any single character						

ab	a followed by b	(**AB**)
(a**|**b**)**	a or b	$(\mathbf{A} \cup \mathbf{B})$
a**?**	zero or one of a	$(\mathbf{A} \cup \varepsilon)$
a*****	zero or more of a	**A***
a**+**	one or more of a	**AA***

- e.g. **`^[\-+]?[0-9]*(\.|\,)?[0-9]+$`**

  General form of decimal number e.g. 9.12 or -9,8 (Europe)

Limitations of Regular Expressions

- Not all languages can be specified by regular expressions
- Even some easy things like
  - Palindromes
  - Strings with equal number of 0's and 1's
- But also more complicated structures in programming languages
  - Matched parentheses
  - Properly formed arithmetic expressions
  - etc.