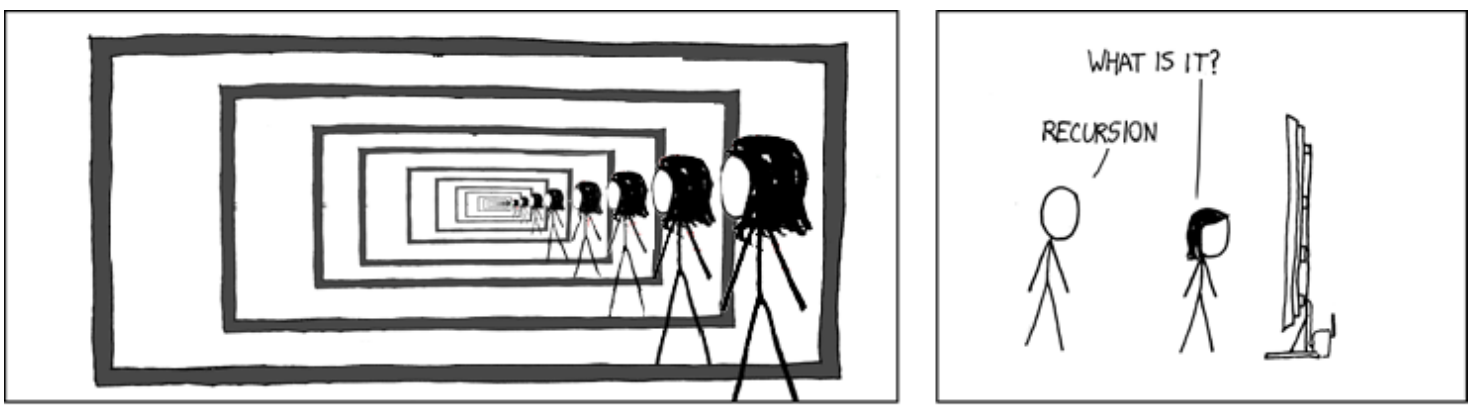
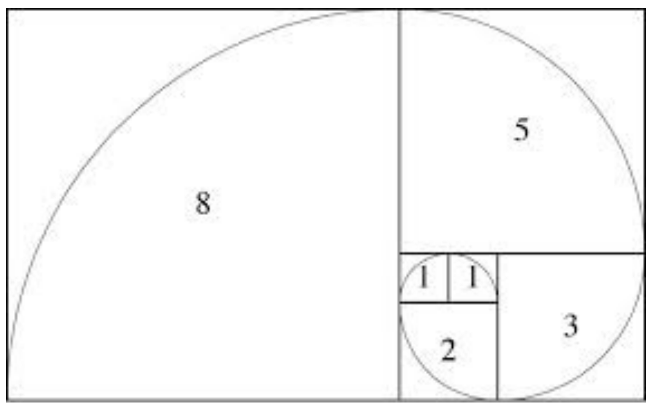
CSE 311: Foundations of Computing

Lecture 18: Recursively Defined Functions & Sets



Last time: Fibonacci Numbers

$f_0 = 0$
$f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ **for all** $n \geq 2$



Last Time: Upper Bound $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, we have $f_j < 2^j$ for every integer j from 0 to k.
4. Inductive Step: $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} < 2^{k+1}}$

   <u>Case k+1 = 1</u>: Then $f_1 = 1 < 2 = 2^1$ so P(k+1) is true here.

   <u>Case k+1 ≥ 2</u>: Then $f_{k+1} = f_k + f_{k-1}$ by definition

   $< 2^k + 2^{k-1}$ by the IH since $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) is true in this case.

   These are the only cases so P(k+1) follows.
5. Therefore by strong induction,

   $f_n < 2^n$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Inductive Proofs with Multiple Base Cases

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."
2. "Base Cases:" Prove $\boxed{P(b), P(b+1), \ldots, P(c)}$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $\boxed{k \geq c}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Inductive Proofs With Multiple Base Cases

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Cases:" Prove $\boxed{P(b), P(b+1), \ldots, P(c)}$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $\boxed{k \geq c}$,

   $P(j)$ is true for every integer j from b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2 - 1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**

$$\boldsymbol{f_0 = 0 \quad f_1 = 1}$$
$$\boldsymbol{f_n = f_{n-1} + f_{n-2}} \text{ for all } \boldsymbol{n \geq 2}$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2 - 1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2 - 1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k**.**

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2\,-\,1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2\ -1}$**". We prove that** P(n) **is true for all integers** n ≥ 2 **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2\,-\,1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** k ≥ 2, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} \geq 2^{(k+1)/2\ -1}}$

No need for cases for the definition here:

$f_{k+1} = f_k + f_{k-1}$ **since** k+1 ≥ 2

Now just want to apply the IH to get P(k) **and** P(k-1)

Problem: Though we can get P(k) **since** k ≥ 2,

k-1 **may only be** 1 **so we can't conclude** P(k-1)

Solution: Separate cases for when k-1=1 **(or** k+1=3**).**

$$\boxed{\begin{array}{l} \boldsymbol{f_0 = 0 \qquad f_1 = 1} \\ \boldsymbol{f_n = f_{n-1} + f_{n-2}} \ \text{ for all } \boldsymbol{n \geq 2} \end{array}}$$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2 - 1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Cases:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2 - 1} = 2^0 = 1$ **so** P(2) **holds**
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ **so** P(3) **holds**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 3$, P(j) **is true for every integer** j **from** 2 **to** k**.**
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} \geq 2^{(k+1)/2 - 1}$

$$\boldsymbol{f_0 = 0 \quad f_1 = 1}$$
$$\boldsymbol{f_n = f_{n-1} + f_{n-2}} \text{ for all } \boldsymbol{n \geq 2}$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be** "$f_n \geq 2^{n/2-1}$". **We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Cases:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **holds**
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$ **so** P(3) **holds**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 3$, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

   We have $f_{k+1} = f_k + f_{k-1}$ **by definition since** $k+1 \geq 2$

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ **by the IH since** $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   so P(k+1) **is true.**
5. **Therefore by strong induction,** $f_n \geq 2^{n/2-1}$ **for all integers** $n \geq 2$.

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid's Algorithm?

We already proved that $f_n \geq 2^{n/2 - 1}$ so $f_{n+1} \geq 2^{(n-1)/2}$

Therefore: if Euclid's Algorithm takes n steps
for $\gcd(a, b)$ with $a \geq b > 0$
then $a \geq 2^{(n-1)/2}$

so $(n-1)/2 \leq \log_2 a$ or $n \leq 1 + 2\log_2 a$
i.e., # of steps ≤ 1 + twice the # of bits in a.

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

An informal way to get the idea: Consider an n **step** gcd **calculation starting with** r_{n+1}=a **and** r_n=b:

$$
\begin{aligned}
r_{n+1} &= q_n r_n + r_{n-1} \\
r_n &= q_{n-1} r_{n-1} + r_{n-2} \\
&\ \ \dots \\
r_3 &= q_2 r_2 + r_1 \\
r_2 &= q_1 r_1
\end{aligned}
$$

For all $k \geq 2$, $r_{k-1} = r_{k+1} \bmod r_k$

Now $r_1 \geq 1$ **and each** q_k **must be** ≥ 1. **If we replace all the** q_K's **by** 1 **and replace** r_1 **by** 1 **, we can only reduce the** r_k's**. After that reduction,** $r_k = f_k$ **for every** k.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\mathrm{gcd}(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n**.**
Let P(n) **be** "gcd(a,b) **with** $a \geq b>0$ **takes** n **steps** $\rightarrow a \geq f_{n+1}$" **for all** $n \geq 1$**.**

__Base Case__: n=1 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 1 **step.**
By assumption, $a \geq b \geq 1 = f_2$ **so** P(1) **holds.**

__Induction Hypothesis__: **Suppose that for some integer** $k \geq 1$, P(j) **is true for all integers** j **s.t.** $1 \leq j \leq k$

__Inductive Step__: **We want to show:** **if** gcd(a,b) **with** $a \geq b > 0$ **takes** k+1 **steps, then** $a \geq f_{k+2}$.

Running time of Euclid's algorithm

Induction Hypothesis: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: $\boxed{\textbf{Goal: if } \gcd(a,b) \textbf{ with } a \geq b>0 \textbf{ takes } k+1 \textbf{ steps, then } a \geq f_{k+2}.}$

Now if $k+1=2$, **then Euclid's algorithm on** a **and** b **can be written as**

$$a = q_2 b + r_1$$
$$b = q_1 r_1$$

and $r_1 > 0$.

Also, since $a \geq b > 0$, **we must have** $q_2 \geq 1$ and $b \geq 1$.

So $a = q_2 b + r_1 \geq b + r_1 \geq 1+1 = 2 = f_3 = f_{k+2}$ as required.

Running time of Euclid's algorithm

Induction Hypothesis: Suppose that for some integer $k \geq 1$, $P(j)$ is true for all integers j s.t. $1 \leq j \leq k$

Inductive Step: $\boxed{\text{Goal: if } \gcd(a,b) \text{ with } a \geq b>0 \text{ takes } k+1 \text{ steps, then } a \geq f_{k+2}.}$

Next suppose that $k+1 \geq 3$ so for the first 3 steps of Euclid's algorithm on a and b we have

$$a = q_{k+1}\, b + r_k$$
$$b = q_k\, r_k + r_{k-1}$$
$$r_k = q_{k-1}\, r_{k-1} + r_{k-2}$$

and there are k-2 more steps after this. Note that this means that the $\gcd(b, r_k)$ takes k steps and $\gcd(r_k, r_{k-1})$ takes k-1 steps.

So since $k, k\text{-}1 \geq 1$, by the IH we have $b \geq f_{k+1}$ and $r_k \geq f_k$.

Also, since $a \geq b$, we must have $q_{k+1} \geq 1$.

So $a = q_{k+1}b + r_k \geq b + r_k \geq f_{k+1}+ f_k= f_{k+2}$ as required. ■

Last time: Recursive definitions of functions

- $0! = 1;\;\; (n+1)! = (n+1) \cdot n!$ **for all** $n \geq 0$**.**
- $F(0) = 0;\;\; F(n+1) = F(n) + 1$ **for all** $n \geq 0$**.**
- $G(0) = 1;\;\; G(n+1) = 2 \cdot G(n)$ **for all** $n \geq 0$**.**
- $H(0) = 1;\;\; H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$**.**

Last time: Recursive definitions of functions

- Recursive functions allow general computation
  - saw examples not expressible with simple expressions
- So far, we have considered only simple data
  - inputs and outputs were just integers
- We need general data as well…
  - these will also be described *recursively*
  - will allow us to describe data of real programs

    e.g., strings, lists, trees, expressions, propositions, …
- We'll start simple: sets of numbers

Recursive Definitions of Sets (Data)

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Recursive Definition of Sets

Recursive definition of set S

- **Basis Step:** $0 \in S$
- **Recursive Step: If** $x \in S$**, then** $x + 2 \in S$
- **Exclusion Rule: Every element in** S **follows from the basis step and a finite number of recursive steps.**

We need the exclusion rule because otherwise $S=\mathbb{N}$ would satisfy the other two parts. However, we won't always write it down on these slides.

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Basis: $(0, 0) \in S$, $(1, 1) \in S$

Recursive: If $(n-1, x) \in S$ and $(n, y) \in S$, then $(n+1, x + y) \in S$. ?

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Basis: $(0, 0) \in S$, $(1, 1) \in S$

Recursive: If $(n\text{-}1, x) \in S$ and $(n, y) \in S$, then $(n+1, x + y) \in S$.

Fibonacci numbers

Last time: Recursive definitions of functions

- Before, we considered only simple data
  - inputs and outputs were just integers

- Proved facts about those functions with **induction**
  - $n! \leq n^n$
  - $f_n < 2^n$ and $f_n \geq 2^{n/2-1}$

- How do we prove facts about functions that work with more complex (recursively defined) data?
  - we need a more sophisticated form of induction

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: **Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step***

Inductive Hypothesis: **Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step***

Inductive Step: **Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis**

Conclude **that $\forall\, x \in S, P(x)$**

Structural Induction vs. Ordinary Induction

Structural induction follows from ordinary induction:

Define $Q(n)$ to be "for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true."

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Using Structural Induction

- Let S be given by…
  - **Basis:** $6 \in S$; $15 \in S$
  - **Recursive:** if $x, y \in S$ then $x + y \in S$.

Claim: Every element of S is divisible by 3.

Claim: Every element of S is divisible by 3.

1. Let P(x) be "$3|x$". We prove that P(x) is true for all $x \in S$ by structural induction.

Basis: $6 \in S$; $15 \in S$

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

1. **Let** P(x) **be** "3|x". **We prove that** P(x) **is true for all** x ∈ S **by structural induction.**
2. **Base Case:** 3|6 **and** 3|15 **so** P(6) **and** P(15) **are true**

Basis: $6 \in S$; $15 \in S$

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

1. **Let** P(x) **be** "3|x". **We prove that** P(x) **is true for all** x ∈ S **by structural induction.**
2. **Base Case:** 3|6 **and** 3|15 **so** P(6) **and** P(15) **are true**
3. **Inductive Hypothesis: Suppose that** P(x) **and** P(y) **are true for some arbitrary** x,y ∈ S
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(x+y)}$

Basis: $6 \in S$; $15 \in S$

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

1. **Let** P(x) **be** "3|x". **We prove that** P(x) **is true for all** x ∈ S **by structural induction.**
2. **Base Case:** 3|6 **and** 3|15 **so** P(6) **and** P(15) **are true**
3. **Inductive Hypothesis: Suppose that** P(x) **and** P(y) **are true for some arbitrary** x,y ∈ S
4. **Inductive Step:** **Goal: Show** P(x+y)

   Since P(x) **is true,** 3|x **and so** x=3m **for some integer** m **and since** P(y) **is true,** 3|y **and so** y=3n **for some integer** n.
   Therefore x+y=3m+3n=3(m+n) **and thus** 3|(x+y).

   Hence P(x+y) **is true.**
5. **Therefore by induction** 3|x **for all** x ∈ S.

Basis: $6 \in S$; $15 \in S$

Recursive: if $x, y \in S$ then $x + y \in S$

Using Structural Induction

- Let T be given by…
  - **Basis:** $6 \in T$; $15 \in T$
  - **Recursive:** if $x \in T$, then $x + 6 \in T$ and $x + 15 \in T$

- Two base cases and two *recursive* cases

Claim: Every element of T is also in S.

Claim: Every element of S is divisible by 3.

1. Let P(x) be "x ∈ S". We prove that P(x) is true for all x ∈ T by structural induction.

Basis: $6 \in S$; $15 \in S$	**Basis:** $6 \in T$; $15 \in T$
Recursive: if $x, y \in S$, then $x + y \in S$	**Recursive:** if $x \in T$, then $x + 6 \in T$ and $x + 15 \in T$

Claim: Every element of S is divisible by 3.

1. **Let** P(x) **be** "x ∈ S". **We prove that** P(x) **is true for all** x ∈ T **by structural induction.**
2. **Base Case:** 6 ∈ S **and** 15 ∈ S **so** P(6) **and** P(15) **are true**

Basis: $6 \in S$; $15 \in S$	**Basis:** $6 \in T$; $15 \in T$
Recursive: if $x, y \in S$, then $x + y \in S$	**Recursive:** if $x \in T$, then $x + 6 \in T$ and $x + 15 \in T$

Claim: Every element of S is divisible by 3.

1. Let P(x) be "x ∈ S". We prove that P(x) is true for all x ∈ T by structural induction.
2. Base Case: 6 ∈ S and 15 ∈ S so P(6) and P(15) are true
3. Inductive Hypothesis: Suppose that P(x) is true for some arbitrary x ∈ T

Basis: $6 \in S$; $15 \in S$	**Basis:** $6 \in T$; $15 \in T$
Recursive: if $x, y \in S$, then $x + y \in S$	**Recursive:** if $x \in T$, then $x + 6 \in T$ and $x + 15 \in T$

Claim: Every element of S is divisible by 3.

1. **Let** P(x) **be** "x ∈ S". **We prove that** P(x) **is true for all** x ∈ T **by structural induction.**
2. **Base Case:** 6 ∈ S **and** 15 ∈ S **so** P(6) **and** P(15) **are true**
3. **Inductive Hypothesis: Suppose that** P(x) **is true for some arbitrary** x ∈ T
4. **Inductive Step:** **Goal: Show** P(x+6) **and** P(x+15)

Basis: $6 \in S$; $15 \in S$ **Recursive:** if $x, y \in S$, then $x + y \in S$	**Basis:** $6 \in T$; $15 \in T$ **Recursive:** if $x \in T$, then $x + 6 \in T$ and $x + 15 \in T$

Claim: Every element of S is divisible by 3.

1. Let P(x) be "x ∈ S". We prove that P(x) is true for all x ∈ T by structural induction.
2. Base Case: 6 ∈ S and 15 ∈ S so P(6) and P(15) are true
3. Inductive Hypothesis: Suppose that P(x) is true for some arbitrary x ∈ T
4. Inductive Step: $\boxed{\text{Goal: Show P(x+6) and P(x+15)}}$

   Since P(x) holds, we have x ∈ S. From the recursive step of S, we can see that x + 6 ∈ S, so P(x+6) is true, and we can see that x + 15 ∈ S, so P(x+15) is true.

Basis: $6 \in S$; $15 \in S$	**Basis:** $6 \in T$; $15 \in T$
Recursive: if $x, y \in S$, then $x + y \in S$	**Recursive:** if $x \in T$, then $x + 6 \in T$ and $x + 15 \in T$

Claim: Every element of S is divisible by 3.

1. Let P(x) be "x ∈ S". We prove that P(x) is true for all x ∈ T by structural induction.
2. Base Case: 6 ∈ S and 15 ∈ S so P(6) and P(15) are true
3. Inductive Hypothesis: Suppose that P(x) is true for some arbitrary x ∈ T
4. Inductive Step: $\boxed{\text{Goal: Show P(x+6) and P(x+15)}}$

   Since P(x) holds, we have x ∈ S. From the recursive step of S, we can see that x + 6 ∈ S, so P(x+6) is true, and we can see that x + 15 ∈ S, so P(x+15) is true.
5. Therefore P(x) for all x ∈ T by induction.

Basis: $6 \in S$; $15 \in S$	**Basis:** $6 \in T$; $15 \in T$
Recursive: if $x, y \in S$, then $x + y \in S$	**Recursive:** if $x \in T$, then $x + 6 \in T$ and $x + 15 \in T$