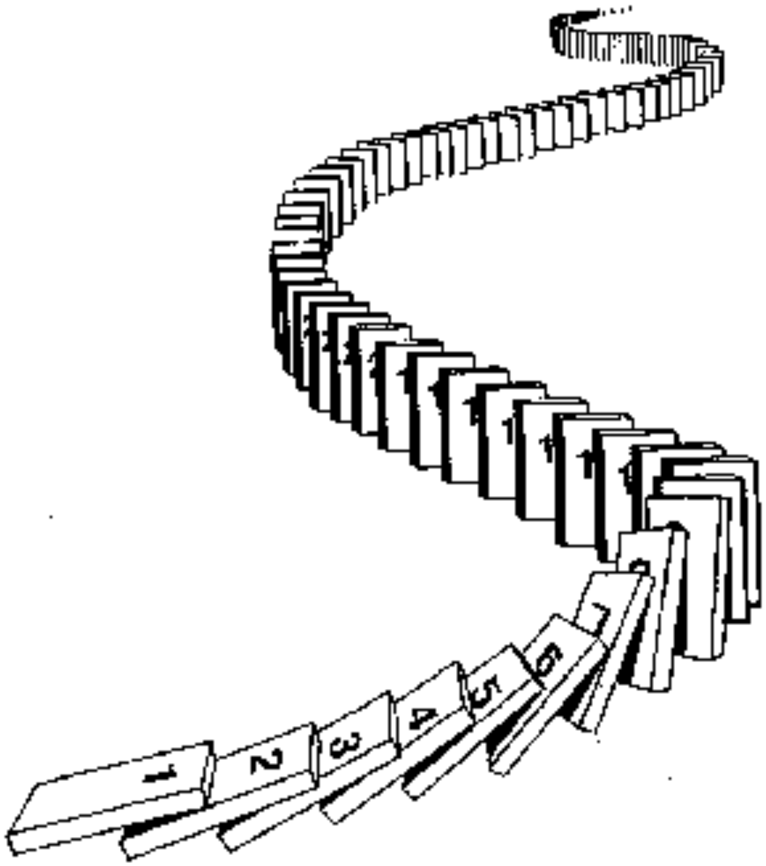
CSE 311: Foundations of Computing

Lecture 16: Induction



Last Time: New Inference Rule

Domain: Natural Numbers

$$\frac{P(0) \qquad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n\ P(n)}$$

Last Time: Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$P(0) \xrightarrow{P(0)\to P(1)} P(1) \xrightarrow{P(1)\to P(2)} P(2) \xrightarrow{P(2)\to P(3)} P(3) \xrightarrow{P(3)\to P(4)} P(4) \xrightarrow{P(4)\to P(5)} P(5)$$

First, we have P(0).
Since P(n) ⟶ P(n+1) for all n, we have P(0) ⟶ P(1).
    Since P(0) is true and P(0) ⟶ P(1), by Modus Ponens, P(1) is true.
Since P(n) ⟶ P(n+1) for all n, we have P(1) ⟶ P(2).
    Since P(1) is true and P(1) ⟶ P(2), by Modus Ponens, P(2) is true.

Last Time: Translating to an English Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

1. Prove P(0) — **Base Case**
2. Let k be an arbitrary integer ≥ 0 — **Inductive Hypothesis**
    3.1. Suppose that P(k) is true
    3.2. ... — **Inductive Step**
    3.3. Prove P(k+1) is true

3.	$P(k) \rightarrow P(k+1)$	Direct Proof Rule
4.	$\forall k\ (P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5.	$\forall n\ P(n)$	Induction: 1, 4

Conclusion

Last Time: Inductive Proofs In 5 Easy Steps

1. **“Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq 0$ by induction.”**
2. **“Base Case:” Prove $P(0)$**
3. **“Inductive Hypothesis:**

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$”
4. **“Inductive Step:” Prove that $P(k+1)$ is true:**

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don’t assume $P(k+1)$!!)
5. **“Conclusion: $P(n)$ is true for all integers $n \geq 0$”**

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 \quad = \quad 1$
- $1 + 2 \quad = \quad 3$
- $1 + 2 + 4 \quad = \quad 7$
- $1 + 2 + 4 + 8 \quad = \quad 15$
- $1 + 2 + 4 + 8 + 16 \quad = \quad 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

Good that we have induction!

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

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1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $\boxed{2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1}$**.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
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3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

$$2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1 \quad \textbf{by IH}$$

Adding 2^{k+1} **to both sides, we get:**

$$2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$**.**

So, we have $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$**, which is exactly** P(k+1)**.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

   We can calculate

$$
\begin{aligned}
2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0 + 2^1 + \ldots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} \qquad \textbf{by the IH} \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1,
\end{aligned}
$$

   which is exactly P(k+1)**.**

Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**
3. **Induction Hypothesis: Suppose that** $P(k)$ **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
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2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0+2^1+ \ldots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} && \textbf{by the IH} \\
&= 2(2^{k+1}) - 1 \\
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\end{aligned}
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   which is exactly P(k+1).

5. **Thus** $P(n)$ **is true for all** $n \in \mathbb{N}$**, by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

Summation Notation

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + 3 + \ldots + n$$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**

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2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**. I.e., suppose** 1 + 2 + …+ k = k(k+1)/2

"some" or "an" not <u>any</u>!

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

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4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** 1 + 2 + …+ k+ (k+1) = (k+1)(k+2)/2

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

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4. **Induction Step:**

$$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= (1 + 2 + \ldots + k) + (k+1) \\ &= k(k+1)/2 + (k+1) \quad \textbf{by IH} \\ &= (k+1)(k/2 + 1) \\ &= (k+1)(k+2)/2 \end{aligned}$$

   So, we have shown 1 + 2 + … + k + (k+1) = (k+1)(k+2)/2**,**

   which is exactly P(k+1)**.**

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

Induction: Changing the start line

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b?

- Define predicate $Q(k) = P(k+b)$ for all k.
  - Then $\forall n\, Q(n) \equiv \forall n \geq b\;\; P(n)$

- Ordinary induction for Q:
  - Prove $Q(0) \equiv P(b)$
  - Prove
    $\forall k\, \big(Q(k) \longrightarrow Q(k+1)\big) \equiv \forall k \geq b \big(P(k) \longrightarrow P(k+1)\big)$

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $k \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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1. **Let** P(n) **be** “$3^n \geq n^2+3$”. **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let $P(n)$ be "$3^n \geq n^2+3$". We will show $P(n)$ is true for all integers $n \geq 2$ by induction.
2. Base Case ($n=2$): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ so $P(2)$ is true.
3. Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 2$. I.e., suppose $3^k \geq k^2+3$.

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   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 3^{k+1} \geq (k+1)^2+3}$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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$$\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 3^{k+1} \geq (k+1)^2+3 = k^2+2k+4}$$

$$\begin{aligned} 3^{k+1} &= 3(3^k) \\ &\geq 3(k^2+3) \text{ by the IH} \\ &= 3k^2+9 \\ &= k^2+2k^2+9 \\ &\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 1. \end{aligned}$$

Therefore P(k+1) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**. I.e., suppose** $3^k \geq k^2+3$**.**
4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

   $3^{k+1} = 3(3^k)$

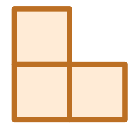
   $\geq 3(k^2+3)$ **by the IH**

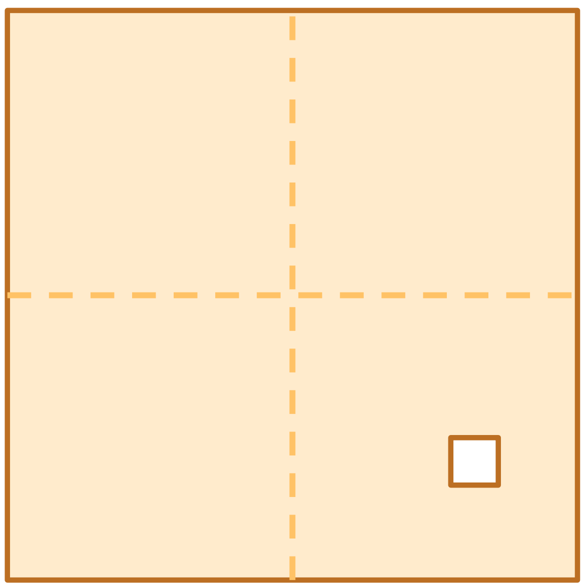
   $= k^2+2k^2+9$

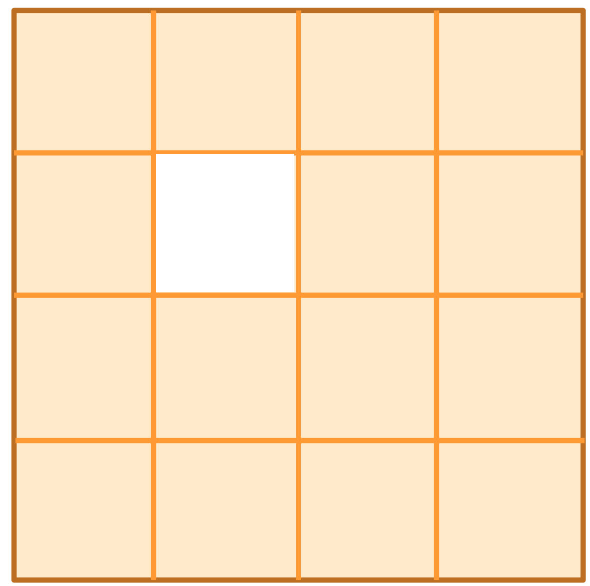
   $\geq k^2+2k+4 = (k+1)^2+3$ **since** $k \geq 1$.

   Therefore P(k+1) **is true.**
5. **Thus** P(n) **is true for all integers** $n \geq 2$**, by induction.**

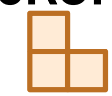
Checkerboard Tiling

- Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 





Checkerboard Tiling

1. Let $P(n)$ be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with  .
   We prove $P(n)$ for all $n \geq 1$ by induction on n.

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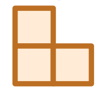
1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
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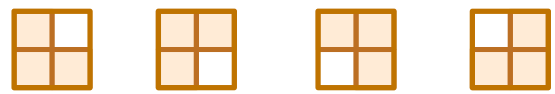
2. **Base Case:** $n=1$ 

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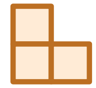
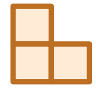
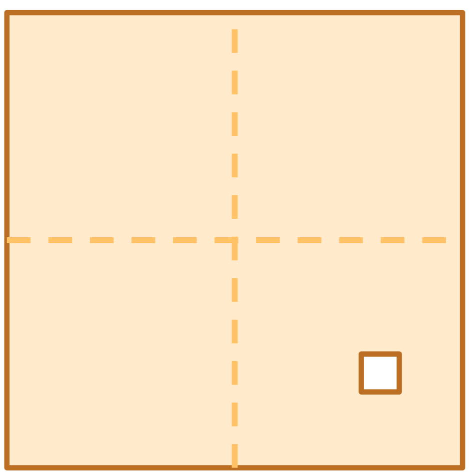
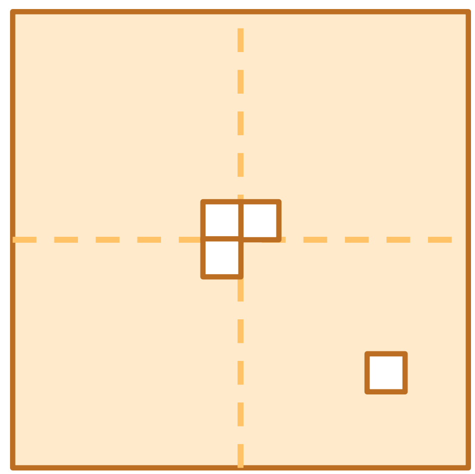
1. Let $P(n)$ be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with  .
   We prove $P(n)$ for all $n \geq 1$ by induction on n.
2. Base Case: $n=1$ 
3. Inductive Hypothesis: Assume $P(k)$ for some arbitrary integer $k \geq 1$

Checkerboard Tiling

1. **Let** P(n) **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove P(n) **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** n=1 
3. **Inductive Hypothesis: Assume** P(k) **for some arbitrary integer** $k \geq 1$
4. **Inductive Step: Prove** P(k+1)





Apply IH to each quadrant then fill with extra tile.