Proofs a Subset Relationship

\[ A ::= \{x : P(x)\} \quad \text{B ::= \{x : Q(x)\}} \]

Let \( x \) be arbitrary

1.1. \( x \in A \) \quad \text{Assumption}
1.2. \( P(x) \) \quad \text{Def of A}
1.8. \( Q(x) \)
1.9. \( x \in B \) \quad \text{Def of B}

1. \( x \in A \rightarrow x \in B \) \quad \text{Direct Proof}
2. \( \forall x (x \in A \rightarrow x \in B) \) \quad \text{Intro} \ \forall: 1
3. \( A \subseteq B \) \quad \text{Def of Subset: 2}
Prove that \( A \subseteq B \).

**Proof:** Let \( x \) be an arbitrary object. Suppose that \( x \in A \). By definition, this means \( P(x) \).

... 

Thus, we have \( Q(x) \). By definition, this means \( x \in B \). Since \( x \) was arbitrary, we have shown, by definition, that \( A \subseteq B \).
De Morgan’s Laws

\[ A \cup B = \bar{A} \cap \bar{B} \]

\[ A \cap B = \bar{A} \cup \bar{B} \]
De Morgan’s Laws

Prove that \((A \cup B)^C = A^C \cap B^C\)

Formally, prove \(\forall x \ (x \in (A \cup B)^C \iff x \in A^C \cap B^C)\)

Proof: Let \(x\) be an arbitrary object.

Since \(x\) was arbitrary, we have shown, by definition, that \((A \cup B)^C = A^C \cap B^C\).
De Morgan’s Laws

Formally, prove $\forall x \ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

1. Let $x$ be arbitrary
   2.1. $x \in (A \cup B)^C$ Assumption
   ...
   2.3. $x \in A^C \cap B^C$
2. $x \in (A \cup B)^C \rightarrow x \in A^C \cap B^C$ Direct Proof
   3.1. $x \in A^C \cap B^C$ Assumption
   ...
   3.3. $x \in (A \cup B)^C$
3. $x \in A^C \cap B^C \rightarrow x \in (A \cup B)^C$ Direct Proof
   4. $(x \in (A \cup B)^C \rightarrow x \in A^C \cap B^C) \land (x \in A^C \cap B^C \rightarrow x \in (A \cup B)^C)$ Intro $\land$: 2, 3
5. $x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C$ Biconditional: 4
6. $\forall x \ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$ Intro $\forall$: 1-5
De Morgan’s Laws

Prove that \((A \cup B)^C = A^C \cap B^C\)
Formally, prove \(\forall x \ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)\)

Proof: Let \(x\) be an arbitrary object.
Suppose \(x \in (A \cup B)^C\).
...

Thus, we have \(x \in A^C \cap B^C\).
De Morgan’s Laws

Prove that \((A \cup B)^C = A^C \cap B^C\)

Formally, prove \(\forall x \ (x \in (A \cup B)^C \iff x \in A^C \cap B^C)\)

Proof: Let \(x\) be an arbitrary object.
Suppose \(x \in (A \cup B)^C\). Then, by the definition of complement, we have \(\neg(x \in A \cup B)\).

... 

Thus, we have \(x \in A^C \cap B^C\).
De Morgan’s Laws

Prove that $(A \cup B)^C = A^C \cap B^C$
Formally, prove $\forall x (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let $x$ be an arbitrary object.
Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \lor x \in B)$.

Thus, we have $x \in A^C \cap B^C$. 
De Morgan’s Laws

Prove that \((A \cup B)^C = A^C \cap B^C\)

Formally, prove \(\forall x \ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)\)

**Proof:** Let \(x\) be an arbitrary object.

Suppose \(x \in (A \cup B)^C\). Then, by the definition of complement, we have \(\neg (x \in A \cup B)\). The latter says, by the definition of union, that \(\neg (x \in A \lor x \in B)\).

... 

Thus, \(x \in A^C\) and \(x \in B^C\), so we we have \(x \in A^C \cap B^C\) by the definition of intersection.
De Morgan’s Laws

Prove that \((A \cup B)^C = A^C \cap B^C\)

Formally, prove \(\forall x (x \in (A \cup B)^C \iff x \in A^C \cap B^C)\)

**Proof:** Let \(x\) be an arbitrary object.

Suppose \(x \in (A \cup B)^C\). Then, by the definition of complement, we have \(\neg(x \in A \cup B)\). The latter says, by the definition of union, that \(\neg(x \in A \lor x \in B)\).

... 

Thus, \(\neg(x \in A)\) and \(\neg(x \in B)\), so \(x \in A^C\) and \(x \in B^C\) by the definition of complement, and we can see that \(x \in A^C \cap B^C\) by the definition of intersection.
De Morgan’s Laws

Prove that \((A \cup B)^C = A^C \cap B^C\)
Formally, prove \(\forall x \ (x \in (A \cup B)^C \iff x \in A^C \cap B^C)\)

Proof: Let \(x\) be an arbitrary object.
Suppose \(x \in (A \cup B)^C\). Then, by the definition of complement, we have \(\neg (x \in A \cup B)\). The latter says, by the definition of union, that \(\neg (x \in A \lor x \in B)\), or equivalently \(\neg (x \in A) \land \neg (x \in B)\) by De Morgan’s law. Thus, we have \(x \in A^C\) and \(x \in B^C\) by the definition of compliment, and we can see that \(x \in A^C \cap B^C\) by the definition of intersection.

Proof technique:
To show \(C = D\) show
\(x \in C \rightarrow x \in D\) and
\(x \in D \rightarrow x \in C\)
De Morgan’s Laws

Prove that \((A \cup B)^C = A^C \cap B^C\)
Formally, prove \(\forall x \ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)\)

Proof: Let \(x\) be an arbitrary object.
Suppose \(x \in (A \cup B)^C\). Then, \(x \in A^C \cap B^C\).
Suppose \(x \in A^C \cap B^C\). Then, by the definition of intersection, we have \(x \in A^C\) and \(x \in B^C\). That is, we have \(\neg(x \in A) \land \neg(x \in B)\), which is equivalent to \(\neg(x \in A \lor x \in B)\) by De Morgan’s law. The last is equivalent to \(\neg(x \in A \cup B)\), by the definition of union, so we have shown \(x \in (A \cup B)^C\), by the definition of complement.
Proofs About Set Equality

A lot of *repetitive* work to show $\rightarrow$ and $\leftarrow$.

Do we have a way to prove $\leftrightarrow$ directly?

Recall that $A \equiv B$ and $(A \leftrightarrow B) \equiv T$ are the same.

We can use an equivalence chain to prove that a biconditional holds.
De Morgan’s Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x \ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let $x$ be an arbitrary object.

The stated biconditional holds since:

\[
x \in (A \cup B)^C \equiv \neg (x \in A \cup B) \quad \text{Def of Comp}
\]

\[
\equiv \neg (x \in A \lor x \in B) \quad \text{Def of Union}
\]

\[
\equiv \neg (x \in A) \land \neg (x \in B) \quad \text{De Morgan}
\]

\[
\equiv x \in A^C \land x \in B^C \quad \text{Def of Comp}
\]

\[
\equiv x \in A^C \cap B^C \quad \text{Def of Union}
\]

Since $x$ was arbitrary, we have shown the sets are equal. ■
Distributive Laws

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
It’s Propositional Logic Again!

**Meta-Theorem:** Translate any Propositional Logic equivalence into “=” relationship between sets by replacing $\cup$ with $\lor$, $\cap$ with $\land$, and $\cdot^C$ with $\neg$.

\[
\begin{align*}
\overline{A \cup B} &= \overline{A} \cap \overline{B} \\
\overline{A \cap B} &= \overline{A} \cup \overline{B}
\end{align*}
\]

\[
\begin{align*}
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C)
\end{align*}
\]
Proving Sets are Equal

Meta-Theorem: Translate any Propositional Logic equivalence into “=” relationship between sets by replacing $\cup$ with $\lor$, $\cap$ with $\land$, and $\cdot^C$ with $\neg$.

“Proof”: Let $x$ be an arbitrary object.
The stated bi-condition holds since:

$x \in \text{left side} \equiv \text{replace set ops with propositional logic}$

$\equiv \text{apply Propositional Logic equivalence}$

$\equiv \text{replace propositional logic with set ops}$

$\equiv x \in \text{right side}$

Since $x$ was arbitrary, we have shown the sets are equal. ■
Power Set

• Power Set of a set $A$ = set of all subsets of $A$

\[ \mathcal{P}(A) ::= \{ B : B \subseteq A \} \]

• e.g., let $\text{Days}=\{M,W,F\}$ and consider all the possible sets of days in a week you could ask a question in class

\[ \mathcal{P}(\text{Days})=? \]

\[ \mathcal{P}(\emptyset)=? \]
Power Set

- Power Set of a set $A = \text{set of all subsets of } A$

$$\mathcal{P}(A) ::= \{B : B \subseteq A \}$$

- e.g., let $\text{Days}=\{M,W,F\}$ and consider all the possible sets of days in a week you could ask a question in class

$$\mathcal{P}(\text{Days})=\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \emptyset\}$$

$$\mathcal{P}(\emptyset)=?$$
Power Set

• Power Set of a set $A = \text{set of all subsets of } A$

$$\mathcal{P}(A) ::= \{B : B \subseteq A \}$$

• e.g., let $\text{Days} = \{\text{M, W, F}\}$ and consider all the possible sets of days in a week you could ask a question in class

$$\mathcal{P}($\text{Days}$) = \{\{\text{M, W, F}\}, \{\text{M, W}\}, \{\text{M, F}\}, \{\text{W, F}\}, \{\text{M}\}, \{\text{W}\}, \{\text{F}\}, \emptyset\}$$

$$\mathcal{P}(\emptyset) = \{\emptyset\} \neq \emptyset$$
Cartesian Product

\[ A \times B := \{ x : \exists a \in A, \exists b \in B \ (x = (a, b)) \} \]

\( \mathbb{R} \times \mathbb{R} \) is the real plane. You’ve seen ordered pairs before.

These are just for arbitrary sets.

\( \mathbb{Z} \times \mathbb{Z} \) is “the set of all pairs of integers”

If \( A = \{1, 2\} \), \( B = \{a, b, c\} \), then \( A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\} \).
Cartesian Product

\[ A \times B := \{ x : \exists a \in A, \exists b \in B \ (x = (a, b)) \} \]

\( \mathbb{R} \times \mathbb{R} \) is the real plane. You’ve seen ordered pairs before.

These are just for arbitrary sets.

\( \mathbb{Z} \times \mathbb{Z} \) is “the set of all pairs of integers”

If \( A = \{1, 2\} \), \( B = \{a, b, c\} \), then \( A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\} \).

What is \( A \times \emptyset \)?
**Cartesian Product**

\[ A \times B ::= \{ x : \exists a \in A, \exists b \in B \ (x = (a, b)) \} \]

\( \mathbb{R} \times \mathbb{R} \) is the real plane. You’ve seen ordered pairs before.

These are just for arbitrary sets.

\( \mathbb{Z} \times \mathbb{Z} \) is “the set of all pairs of integers”

If \( A = \{1, 2\}, \ B = \{a, b, c\}, \) then \( A \times B = \{(1,a), (1,b), (1,c), \)
\( (2,a), (2,b), (2,c)\}\).

\[ A \times \emptyset = \{(a, b) : a \in A \ \land \ b \in \emptyset\} = \{(a, b) : a \in A \ \land \ F\} = \emptyset \]
Russell’s Paradox

\[ S ::= \{ x : x \notin x \} \]

Suppose that \( S \in S \).
Russell’s Paradox

\[ S ::= \{ x : x \not\in x \} \]

Suppose that \( S \in S \). Then, by the definition of \( S \), \( S \not\in S \), but that’s a contradiction.

Suppose that \( S \not\in S \). Then, by the definition of \( S \), \( S \in S \), but that’s a contradiction too.

This is reminiscent of the truth value of the statement “This statement is false.”
More Logic
Induction
Mathematical Induction

Method for proving statements about all natural numbers

- A new logical inference rule!
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily

- Particularly useful for reasoning about programs!

  ```java
  for (int i=0; i < n; n++) { ... }
  ```
  - Show P(i) holds after i times through the loop
Prove $\forall a, b, m > 0 \forall k \in \mathbb{N} \ ((a \equiv_m b) \rightarrow (a^k \equiv_m b^k))$

Let $a, b, m > 0$ be arbitrary. Let $k \in \mathbb{N}$ be arbitrary.  
Suppose that $a \equiv_m b$.

We know $((a \equiv_m b) \land (a \equiv_m b)) \rightarrow (a^2 \equiv_m b^2)$ by multiplying congruences. So, applying this repeatedly, we have:

$((a \equiv_m b) \land (a \equiv_m b)) \rightarrow (a^2 \equiv_m b^2)$
$((a^2 \equiv_m b^2) \land (a \equiv_m b)) \rightarrow (a^3 \equiv_m b^3)$

...$((a^{k-1} \equiv_m b^{k-1}) \land (a \equiv_m b)) \rightarrow (a^k \equiv_m b^k)$

The “...”s is a problem! We don’t have a proof rule that allows us to say “do this over and over”.

But there such a property of the natural numbers!

Domain: Natural Numbers

\[
P(0) \\
\forall k \ (P(k) \rightarrow P(k + 1)) \\
\therefore \forall n \ P(n)
\]
Induction Is A Rule of Inference

Domain: Natural Numbers

\[
P(0)
\]
\[
\forall k \ (P(k) \rightarrow P(k + 1))
\]
\[
\therefore \forall n \ P(n)
\]

How do the givens prove P(3)?
Induction Is A Rule of Inference

Domain: Natural Numbers

\[ \begin{align*}
P(0) \\
\forall k \ (P(k) \rightarrow P(k + 1)) \\
\therefore \forall n \ P(n)
\end{align*} \]

How do the givens prove \( P(5) \)?

First, we have \( P(0) \).
Since \( P(n) \rightarrow P(n+1) \) for all \( n \), we have \( P(0) \rightarrow P(1) \).
Since \( P(0) \) is true and \( P(0) \rightarrow P(1) \), by Modus Ponens, \( P(1) \) is true.
Since \( P(n) \rightarrow P(n+1) \) for all \( n \), we have \( P(1) \rightarrow P(2) \).
Since \( P(1) \) is true and \( P(1) \rightarrow P(2) \), by Modus Ponens, \( P(2) \) is true.
Using The Induction Rule In A Formal Proof

\begin{align*}
P(0) \\
\forall k \ (P(k) \rightarrow P(k + 1)) \\
\therefore \forall n \ P(n)
\end{align*}
Using The Induction Rule In A Formal Proof

\[
P(0) \\
\forall k \ (P(k) \rightarrow P(k + 1)) \\
\therefore \forall n \ P(n)
\]

1. $P(0)$

4. $\forall k \ (P(k) \rightarrow P(k+1))$

5. $\forall n \ P(n)$

Induction: 1, 4
Using The Induction Rule In A Formal Proof

\[
P(0) \\
\forall k (P(k) \rightarrow P(k + 1)) \\
\therefore \forall n P(n)
\]

1. \(P(0)\)
2. Let \(k\) be an arbitrary integer \(\geq 0\)

3. \(P(k) \rightarrow P(k+1)\)
4. \(\forall k (P(k) \rightarrow P(k+1))\) \hspace{1cm} \text{Intro } \forall: 2, 3
5. \(\forall n P(n)\) \hspace{1cm} \text{Induction: 1, 4}
Using The Induction Rule In A Formal Proof

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]
\[ \therefore \forall n \ P(n) \]

1. \( P(0) \)
2. Let \( k \) be an arbitrary integer \( \geq 0 \)
   3.1. \( P(k) \) \hspace{1cm} \text{Assumption}
   3.2. ... 
   3.3. \( P(k+1) \)
3. \( P(k) \rightarrow P(k+1) \) \hspace{1cm} \text{Direct Proof Rule}
4. \( \forall k \ (P(k) \rightarrow P(k+1)) \) \hspace{1cm} \text{Intro \( \forall \): 2, 3}
5. \( \forall n \ P(n) \) \hspace{1cm} \text{Induction: 1, 4}
Translating to an English Proof

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]
\[ \therefore \forall n \ P(n) \]

1. Prove \( P(0) \) [**Base Case**]
2. Let \( k \) be an arbitrary integer \( \geq 0 \)
   3.1. Suppose that \( P(k) \) is true
   3.2. ...
   3.3. Prove \( P(k+1) \) is true [**Inductive Step**]
3. \( P(k) \rightarrow P(k+1) \) [Direct Proof Rule]
4. \( \forall k \ (P(k) \rightarrow P(k+1)) \) [Intro \( \forall \): 2, 3]
5. \( \forall n \ P(n) \) [Induction: 1, 4]

Conclusion
Translating to an English Proof

Induction English Proof Template

[...Define P(n)...]
We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.

Base Case: [...proof of $P(0)$ here...]

Induction Hypothesis:
Suppose that $P(k)$ is true for an arbitrary $k \in \mathbb{N}$.

Induction Step:
[...proof of $P(k + 1)$ here...]

The proof of $P(k + 1)$ must invoke the IH somewhere.

So, the claim is true by induction.
Inductive Proofs In 5 Easy Steps

**Proof:**

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for every $n \geq 0$ by Induction.”

2. “Base Case:” Prove $P(0)$

3. “Inductive Hypothesis:

   Suppose $P(k)$ is true for an arbitrary integer $k \geq 0$”

4. “Inductive Step:” Prove that $P(k + 1)$ is true.

   *Use the goal to figure out what you need.*

   *Make sure you are using I.H. and point out where you are using it. (Don’t assume $P(k + 1)$ !*)

5. “Conclusion: Result follows by induction”
What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

Good that we have induction!
Prove \( 1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1 \)
Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let $P(n)$ be “$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$”. We will show $P(n)$ is true for all natural numbers by induction.
Prove \(1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1\)

1. Let \(P(n)\) be "\(2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1\)". We will show \(P(n)\) is true for all natural numbers by induction.

2. Base Case (\(n=0\)): \(2^0 = 1 = 2 - 1 = 2^{0+1} - 1\) so \(P(0)\) is true.
Prove $1 + 2 + 4 + ... + 2^n = 2^{n+1} - 1$

1. Let $P(n)$ be “$2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1$”. We will show $P(n)$ is true for all natural numbers by induction.

2. Base Case ($n=0$): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so $P(0)$ is true.

3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 0$, i.e., that $2^0 + 2^1 + ... + 2^k = 2^{k+1} - 1$. Adding $2^{k+1}$ to both sides, we get:

$$1 + 2 + ... + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

So, we have $1 + 2 + ... + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly $P(k+1)$.

5. Thus $P(k)$ is true for all $k \in \mathbb{N}$, by induction.
1. Let \( P(n) \) be “\( 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1 \)”. We will show \( P(n) \) is true for all natural numbers by induction.

2. Base Case (\( n=0 \)): \( 2^0 = 1 = 2 - 1 = 2^{0+1} - 1 \) so \( P(0) \) is true.

3. Induction Hypothesis: Suppose that \( P(k) \) is true for some arbitrary integer \( k \geq 0 \), i.e., that \( 2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1 \).

4. Induction Step:

   Goal: Show \( P(k+1) \), i.e. show \( 2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1 \)
1. Let \( P(n) \) be “\( 2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1 \)”. We will show \( P(n) \) is true for all natural numbers by induction.

2. Base Case \((n=0)\): \( 2^0 = 1 = 2 - 1 = 2^{0+1} - 1 \) so \( P(0) \) is true.

3. Induction Hypothesis: Suppose that \( P(k) \) is true for some arbitrary integer \( k \geq 0 \), i.e., that \( 2^0 + 2^1 + ... + 2^k = 2^{k+1} - 1 \).

4. Induction Step:

\[
2^0 + 2^1 + ... + 2^k = 2^{k+1} - 1 \quad \text{by IH}
\]

Adding \( 2^{k+1} \) to both sides, we get:

\[
2^0 + 2^1 + ... + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1
\]

Note that \( 2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2} \).

So, we have \( 2^0 + 2^1 + ... + 2^k + 2^{k+1} = 2^{k+2} - 1 \), which is exactly \( P(k+1) \).
Let $P(n)$ be “$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$”. We will show $P(n)$ is true for all natural numbers by induction.

2. Base Case ($n=0$): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so $P(0)$ is true.

3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 0$, i.e., that $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.

4. Induction Step:
   We can calculate
   \[
   2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = (2^0+2^1+\ldots+2^k) + 2^{k+1}
   = (2^{k+1} - 1) + 2^{k+1}
   = 2(2^{k+1}) - 1
   = 2^{k+2} - 1,
   \]
   which is exactly $P(k+1)$.

Alternative way of writing the inductive step
Prove \(1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1\)

1. Let \(P(n)\) be “\(2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1\)”. We will show \(P(n)\) is true for all natural numbers by induction.

2. Base Case (\(n=0\)): \(2^0 = 1 = 2 - 1 = 2^{0+1} - 1\) so \(P(0)\) is true.

3. Induction Hypothesis: Suppose that \(P(k)\) is true for some arbitrary integer \(k \geq 0\), i.e., that \(2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1\).

4. Induction Step:
   We can calculate
   \[
   2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = (2^0+2^1+\ldots+2^k) + 2^{k+1}
   = (2^{k+1} - 1) + 2^{k+1}
   = 2(2^{k+1}) - 1
   = 2^{k+2} - 1,
   \]
   which is exactly \(P(k+1)\).

5. Thus \(P(n)\) is true for all \(n \in \mathbb{N}\), by induction.
Prove $1 + 2 + 3 + \ldots + n = n(n + 1)/2$
1. **Let** $P(n)$ be “$0 + 1 + 2 + ... + n = n(n+1)/2$”. **We will show** $P(n)$ **is true for all natural numbers by induction.**
Prove $1 + 2 + 3 + \ldots + n = n(n + 1)/2$

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   Goal: Show $P(k+1)$, i.e. show $1 + 2 + \ldots + k + (k+1) = (k+1)(k+2)/2$
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4. Induction Step:

$$1 + 2 + ... + k + (k+1) = (1 + 2 + ... + k) + (k+1)$$
$$= k(k+1)/2 + (k+1) \text{ by IH}$$
$$= (k+1)(k/2 + 1)$$
$$= (k+1)(k+2)/2$$

So, we have shown $1 + 2 + ... + k + (k+1) = (k+1)(k+2)/2$, which is exactly $P(k+1)$.

5. Thus $P(n)$ is true for all $n \in \mathbb{N}$, by induction.