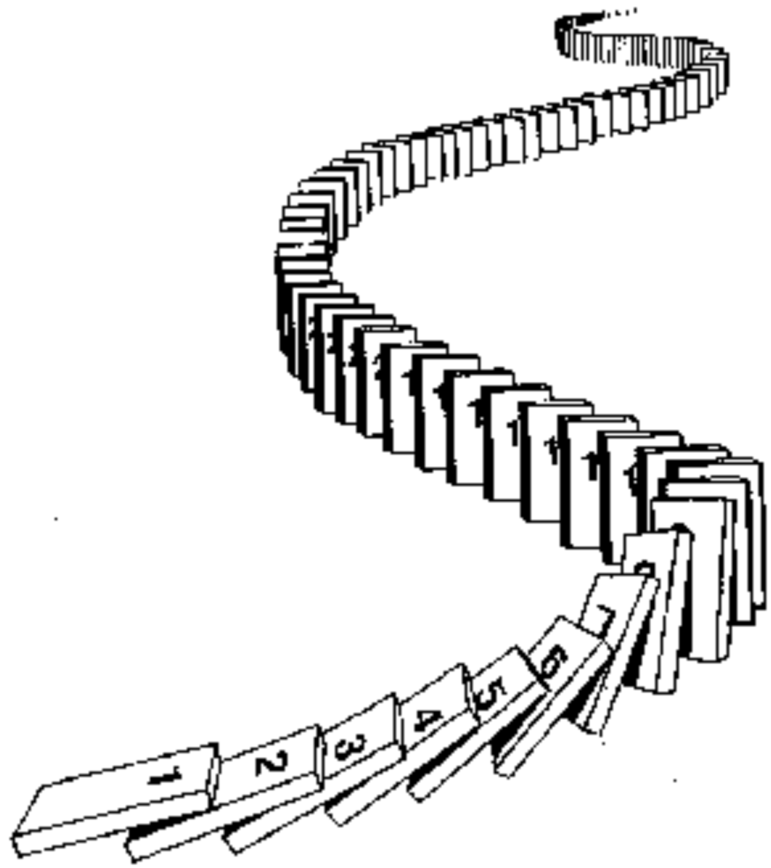
CSE 311: Foundations of Computing

Lecture 15: Set Theory & Induction



Proofs a Subset Relationship

$A ::= \{x : P(x)\}$ $B ::= \{x : Q(x)\}$

Let x be arbitrary		
1.1.	$x \in A$	Assumption
1.2.	$P(x)$	Def of **A**
1.8.	$Q(x)$	
1.9.	$x \in B$	Def of **B**
1.	$x \in A \to x \in B$	Direct Proof
2.	$\forall x\,(x \in A \to x \in B)$	Intro $\forall$: 1
3.	$A \subseteq B$	Def of Subset: 2

Proofs About Sets

$A ::= \{x : P(x)\}$ $\quad$ $B ::= \{x : Q(x)\}$

Prove that $A \subseteq B$**.**

Proof: Let x be an arbitrary object.

Suppose that $x \in A$. By definition, this means $P(x)$.

…

Thus, we have $Q(x)$. By definition, this means $x \in B$.

Since x was arbitrary, we have shown, by definition, that $A \subseteq B$.

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Since x was arbitrary, we have shown, by definition, that $(A \cup B)^C = A^C \cap B^C$.

Proof technique:
To show C = D show
$x \in C \rightarrow x \in D$ and
$x \in D \rightarrow x \in C$

De Morgan's Laws

Formally, prove $\forall x\,(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

1. Let x be arbitrary

    2.1. $x \in (A \cup B)^C$ — Assumption

    ...

    2.3. $x \in A^C \cap B^C$

2. $x \in (A \cup B)^C \rightarrow x \in A^C \cap B^C$ — Direct Proof

    3.1. $x \in A^C \cap B^C$ — Assumption

    ...

    3.3. $x \in (A \cup B)^C$

3. $x \in A^C \cap B^C \rightarrow x \in (A \cup B)^C$ — Direct Proof
4. $(x \in (A \cup B)^C \rightarrow x \in A^C \cap B^C) \wedge (x \in A^C \cap B^C \rightarrow x \in (A \cup B)^C)$ — Intro $\wedge$: 2, 3
5. $x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C$ — Biconditional: 4
6. $\forall x\,(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$ — Intro $\forall$: 1-5

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x} \, (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$.

...

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$.

…

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall$x $(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, $x \in A^C$ and $x \in B^C$, so we we have $x \in A^C \cap B^C$ by the definition of intersection.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall$x $(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, $\neg(x \in A)$ and $\neg(x \in B)$, so $x \in A^C$ and $x \in B^C$ by the definition of compliment, and we can see that $x \in A^C \cap B^C$ by the definition of intersection.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall$x $(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$, or equivalently $\neg(x \in A) \wedge \neg(x \in B)$ by De Morgan's law. Thus, we have $x \in A^C$ and $x \in B^C$ by the definition of compliment, and we can see that $x \in A^C \cap B^C$ by the definition of intersection.

Proof technique:
To show C = D show
$x \in$ C $\rightarrow x \in$ D and
$x \in$ D $\rightarrow x \in$ C

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$.... Then, $x \in A^C \cap B^C$.

Suppose $x \in A^C \cap B^C$. Then, by the definition of intersection, we have $x \in A^C$ and $x \in B^C$. That is, we have $\neg(x \in A) \wedge \neg(x \in B)$, which is equivalent to $\neg(x \in A \vee x \in B)$ by De Morgan's law. The last is equivalent to $\neg(x \in A \cup B)$, by the definition of union, so we have shown $x \in (A \cup B)^C$, by the definition of complement.

Proofs About Set Equality

A lot of *repetitive* work to show $\longrightarrow$ and $\longleftarrow$.

Do we have a way to prove $\leftrightarrow$ directly?

Recall that $A \equiv B$ and $(A \leftrightarrow B) \equiv T$ are the same

We can use an equivalence chain to prove that a biconditional holds.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

The stated biconditional holds since:

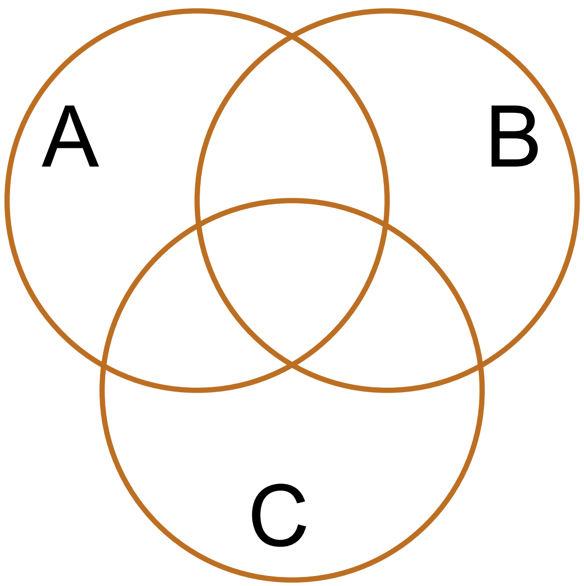
$x \in (A \cup B)^C$	$\equiv \neg(x \in A \cup B)$	Def of Comp
	$\equiv \neg(x \in A \vee x \in B)$	Def of Union
	$\equiv \neg(x \in A) \wedge \neg(x \in B)$	De Morgan
	$\equiv x \in A^C \wedge x \in B^C$	Def of Comp
	$\equiv x \in A^C \cap B^C$	Def of Union

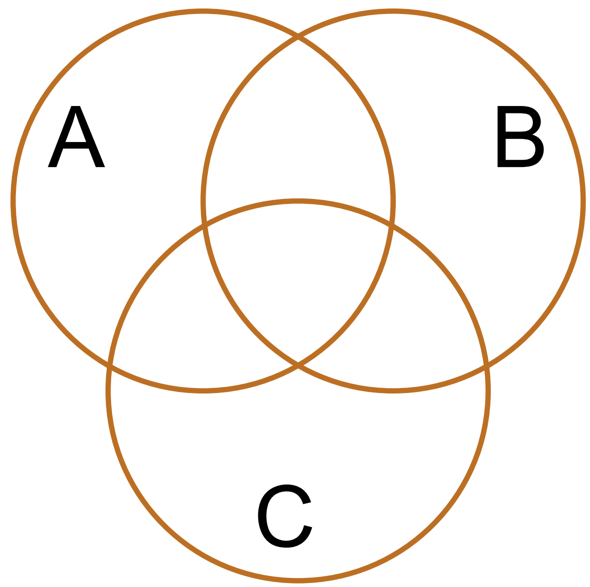
Chains of equivalences are often easier to read like this rather than as English text

Since x was arbitrary, we have shown the sets are equal. ■

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





It's Propositional Logic Again!

Meta-Theorem: Translate any Propositional Logic equivalence into "=" relationship between sets by replacing ∪ with ∨, ∩ with ∧, and $\cdot^{C}$ with ¬.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proving Sets are Equal

Meta-Theorem: Translate any Propositional Logic equivalence into "=" relationship between sets by replacing $\cup$ with $\vee$, $\cap$ with $\wedge$, and $\cdot^{C}$ with $\neg$.

"Proof": Let x be an arbitrary object.

The stated bi-condition holds since:

$x \in$ left side $\quad\equiv$ replace set ops with propositional logic

$\equiv$ apply Propositional Logic equivalence

$\equiv$ replace propositional logic with set ops

$\equiv x \in$ right side

Since x was arbitrary, we have shown the sets are equal. ■

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) ::= \{B : B \subseteq A\}$$

- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$\mathcal{P}$(Days)=?

$\mathcal{P}(\emptyset)$=?

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- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$\mathcal{P}(\text{Days})=\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\}$

$\mathcal{P}(\varnothing)=?$

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) ::= \{B : B \subseteq A\}$$

- e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class

$\mathcal{P}(\text{Days})=\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\}$

$\mathcal{P}(\varnothing)=\{\varnothing\} \neq \varnothing$

Cartesian Product

$$A \times B ::= \{x : \exists a \in A, \exists b \in B \ (x = (a, b)) \}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

Cartesian Product

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If A = {1, 2}, B = {a, b, c}, then A $\times$ B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

What is $A \times \emptyset$?

Cartesian Product

$$A \times B ::= \{x : \exists a \in A, \exists b \in B\ (x = (a, b))\ \}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

$$\boldsymbol{A} \times \emptyset = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \ \wedge\ \boldsymbol{b} \in \emptyset\} = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \ \wedge\ \mathbf{F}\} = \emptyset$$

Russell's Paradox

$$S ::= \{x : x \notin x\}$$

Suppose that $S \in S$**...**

Russell's Paradox

$$S ::= \{x : x \notin x\}$$

Suppose that $S \in S$. Then, by the definition of S, $S \notin S$, but that's a contradiction.

Suppose that $S \notin S$. Then, by the definition of S, $S \in S$, but that's a contradiction too.

This is reminiscent of the truth value of the statement "This statement is false."

More Logic

Induction

Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily

- **Particularly useful for reasoning about programs!**

```
for (int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

Prove $\forall a, b, m > 0\ \forall k \in \mathbb{N}\ ((a \equiv_m b) \rightarrow (a^k \equiv_m b^k))$

Let $a, b, m > 0$ **be arbitrary. Let** $k \in \mathbb{N}$ **be arbitrary.**

Suppose that $a \equiv_m b$**.**

We know $((a \equiv_m b) \wedge (a \equiv_m b)) \rightarrow (a^2 \equiv_m b^2)$ **by multiplying congruences. So, applying this repeatedly, we have:**

$$((a \equiv_m b) \wedge (a \equiv_m b)) \rightarrow (a^2 \equiv_m b^2)$$
$$\big((a^2 \equiv_m b^2) \wedge (a \equiv_m b)\big) \rightarrow (a^3 \equiv_m b^3)$$
$$\ldots$$
$$\big((a^{k-1} \equiv_m b^{k-1}) \wedge (a \equiv_m b)\big) \rightarrow (a^k \equiv_m b^k)$$

The "…"s is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there such a property of the natural numbers!

Domain: Natural Numbers

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(3)?

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$\begin{tikzcd} P(0) \arrow[r, bend left, "P(0)\to P(1)"] & P(1) \arrow[r, bend left, "P(1)\to P(2)"] & P(2) \arrow[r, bend left, "P(2)\to P(3)"] & P(3) \arrow[r, bend left, "P(3)\to P(4)"] & P(4) \arrow[r, bend left, "P(4)\to P(5)"] & P(5) \end{tikzcd}$$

First, we have P(0).

Since P(n) ⟶ P(n+1) for all n, we have P(0) ⟶ P(1).

Since P(0) is true and P(0) ⟶ P(1), by Modus Ponens, P(1) is true.

Since P(n) ⟶ P(n+1) for all n, we have P(1) ⟶ P(2).

Since P(1) is true and P(1) ⟶ P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)

4. $\forall$k (P(k) $\rightarrow$ P(k+1))
5. $\forall$n P(n) Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)
2. Let k be an arbitrary integer ≥ 0

3. $P(k) \rightarrow P(k+1)$
4. $\forall k\ (P(k) \rightarrow P(k+1))$ Intro $\forall$: 2, 3
5. $\forall n\ P(n)$ Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)	
2. Let k be an arbitrary integer ≥ 0	
3.1. P(k)	Assumption
3.2. ...	
3.3. P(k+1)	
3. $P(k) \rightarrow P(k+1)$	Direct Proof Rule
4. $\forall k\ (P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5. $\forall n\ P(n)$	Induction: 1, 4

Translating to an English Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0) **Base Case**

2. Let k be an arbitrary integer ≥ 0
   3.1. Suppose that P(k) is true

Inductive Hypothesis

   3.2. ...
   3.3. Prove P(k+1) is true

Inductive Step

3.	$P(k) \to P(k+1)$	Direct Proof Rule
4.	$\forall k\ (P(k) \to P(k+1))$	Intro $\forall$: 2, 3
5.	$\forall n\ P(n)$	Induction: 1, 4

Conclusion

Translating to an English Proof

1. Prove P(0)		**Base Case**
2. Let k be an arbitrary integer ≥ 0		**Inductive Hypothesis**
3.1. Assume that P(k) is true		
3.2. ...		**Inductive Step**
3.3. Prove P(k+1) is true		
3. $P(k) \to P(k+1)$	Direct Proof Rule	**Conclusion**
4. $\forall k\, (P(k) \to P(k+1))$	Intro $\forall$: 2, 3	
5. $\forall n\, P(n)$	Induction: 1, 4	

Induction English Proof Template

[...Define P(n)...]

We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.

Base Case: *[...proof of $P(0)$ here...]*

Induction Hypothesis:

Suppose that $P(k)$ is true for an arbitrary $k \in \mathbb{N}$.

Induction Step:

[...proof of $P(k+1)$ here...]

*The proof of $P(k+1)$ **must** invoke the IH somewhere.*

So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof:

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Suppose $P(k)$ is true for an arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true.

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 \qquad = \quad 1$
- $1 + 2 \qquad = \quad 3$
- $1 + 2 + 4 \qquad = \quad 7$
- $1 + 2 + 4 + 8 \qquad = \quad 15$
- $1 + 2 + 4 + 8 + 16 \qquad = \quad 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ **but that would literally take forever.**

Good that we have induction!

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $\boxed{2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1}$.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
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3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

Goal: Show P(k+1)**, i.e. show** $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

$$2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1 \quad \textbf{by IH}$$

Adding 2^{k+1} **to both sides, we get:**

$$2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

So, we have $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** P(k+1).

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let P(n) be "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". We will show P(n) is true for all natural numbers by induction.
2. Base Case (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so P(0) is true.
3. Induction Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 0$, i.e., that $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. Induction Step:

   We can calculate

$$
\begin{aligned}
2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0+2^1+ \ldots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} && \text{by the IH} \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1,
\end{aligned}
$$

   which is exactly P(k+1).

Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let P(n) be "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

   We can calculate

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0+2^1+ \ldots + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} && \textbf{by the IH} \\ &= 2(2^{k+1}) - 1 \\ &= 2^{k+2} - 1, \end{aligned}$$

   which is exactly P(k+1).

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

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Goal: Show P(k+1)**, i.e. show** 1 + 2 + …+ k+ (k+1) = (k+1)(k+2)/2

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

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\begin{aligned}
1 + 2 + \ldots + k + (k+1) &= (1 + 2 + \ldots + k) + (k+1) \\
&= k(k+1)/2 + (k+1) \quad \textbf{by IH} \\
&= (k+1)(k/2 + 1) \\
&= (k+1)(k+2)/2
\end{aligned}
$$

   So, we have shown 1 + 2 + … + k + (k+1) = (k+1)(k+2)/2, **which is exactly** P(k+1)**.**

5. **Thus** P(n) **is true for all** n ∈ℕ**, by induction.**