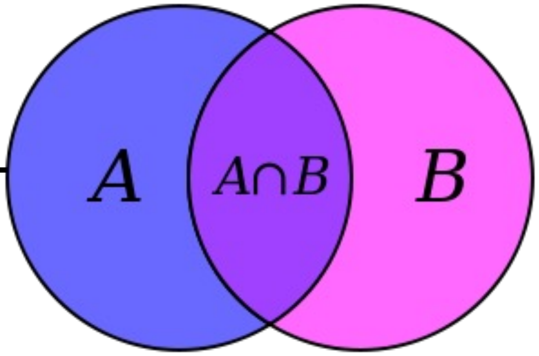
CSE 311: Foundations of Computing

Lecture 14: Set Theory

Sets



Sets are collections of objects called **elements.**

Write $a \in B$ to say that a is an element of set B,
and $a \notin B$ to say that it is not.

Some simple examples
A = {1}
B = {1, 3, 2}
C = {□, 1}
D = {{17}, 17}
E = {1, 2, 7, cat, dog, ∅, α}

Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, $\pi, \sqrt{2}$

[n] is the set **{1, 2, …, n}** when **n** is a natural number

$\varnothing$ = **{}** is the **empty set**; the *only* set with no elements

Sets can be elements of other sets

For example

$A = \{\{1\},\{2\},\{1,2\},\emptyset\}$

$B = \{1,2\}$

Then $B \in A$.

Definitions

- A and B are *equal* if they have the same elements

$$A = B \ ::= \ \forall x \, (x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \ ::= \ \forall x \, (x \in A \rightarrow x \in B)$$

- Notes: $(\boldsymbol{A} = \boldsymbol{B}) \equiv (\boldsymbol{A} \subseteq \boldsymbol{B}) \land (\boldsymbol{B} \subseteq \boldsymbol{A})$

  $A \supseteq B$ means $B \subseteq A$

  $A \subset B$ means $A \subseteq B$

Definition: Equality

A and B are *equal* if they have the same elements

$$A = B \ ::= \ \forall x \, (x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are equal to each other?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \ ::= \ \forall x\, (x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

<u>QUESTIONS</u>

$\varnothing \subseteq A$?
$A \subseteq B$?
$C \subseteq B$?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \;::=\; \forall\, x\, (x \in A \rightarrow x \in B)$$

Note the domain restriction.

We will use a shorthand restriction to a set

$\forall x{\in}A\ (P(x))$ means $\forall x\ (x \in A \rightarrow P(x))$

Restricting all quantified variables improves *clarity*

Sets & Logic

Building Sets from Predicates

Every set S defines a predicate "$x \in S$".

We can also define a set from a predicate P:

$$S ::= \{x : P(x)\}$$

S = the set of all x for which $P(x)$ is true

$$S ::= \{x \in U : P(x)\} = \{x : (x \in U) \land P(x)\}$$

Inference Rules on Sets

S ::= {x : P(x)}

When a set is defined this way, we can reason about it using its definition:

1.	$x \in S$	Given
2.	$P(x)$	Def of S
	…	
8.	$P(y)$	
9.	$y \in S$	Def of S

This will be our **only** inference rule for sets!

Proofs About Sets

$A ::= \{x : P(x)\}$ $B ::= \{x : Q(x)\}$

Suppose we want to prove $A \subseteq B$.

This is a predicate:

$$A \subseteq B ::= \forall x\,(x \in A \rightarrow x \in B)$$

We need to show that is definition holds

Proofs About Sets

$A ::= \{x : P(x)\}$ $B ::= \{x : Q(x)\}$

Let x be arbitrary

1.1. $x \in A$ Assumption

1.9. $x \in B$??

1. $x \in A \rightarrow x \in B$ Direct Proof
2. $\forall x\, (x \in A \rightarrow x \in B)$ Intro $\forall$: 1
3. $A \subseteq B$ Def of Subset: 2

Proofs About Sets

$A ::= \{x : P(x)\}$ $\quad$ $B ::= \{x : Q(x)\}$

Let x be arbitrary

1.1.	$x \in A$	Assumption
1.2.	$P(x)$	Def of **A**
1.8.	$Q(x)$	
1.9.	$x \in B$	Def of **B**

1. $x \in A \rightarrow x \in B$ — Direct Proof
2. $\forall x\, (x \in A \rightarrow x \in B)$ — Intro $\forall$: 1
3. $A \subseteq B$ — Def of Subset: 2

Proofs About Sets

$A ::= \{x : P(x)\}$ $B ::= \{x : Q(x)\}$

Prove that $A \subseteq B$**.**

Proof: Let x be an arbitrary object.

Suppose that $x \in A$. By definition, this means $P(x)$.

…

Thus, we have $Q(x)$. By definition, this means $x \in B$.

Since x was arbitrary, we have shown, by definition, that $A \subseteq B$.

Operations on Sets

Set Operations

$$A \cup B ::= \{\, x : (x \in A) \vee (x \in B)\}$$ **Union**

$$A \cap B ::= \{\, x : (x \in A) \wedge (x \in B)\}$$ **Intersection**

$$A \setminus B ::= \{\, x : (x \in A) \wedge (x \notin B)\}$$ **Set Difference**

A = {1, 2, 3}
B = {3, 5, 6}
C = {3, 4}

QUESTIONS

Using A, B, C and set operations, make…

[6] =

{3} =

{1,2} =

More Set Operations

$$A \oplus B ::= \{\, x : (x \in A) \oplus (x \in B)\}$$

Symmetric Difference

$$\overline{A} = A^C ::= \{\, x : x \in U \wedge x \notin A \,\}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 2, 4, 6}
Universe:
U = {1, 2, 3, 4, 5, 6}

$A \oplus B$ = {3, 4, 6}
$\overline{A}$ = {4,5,6}

Set Complement

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It's remarkable that as recently as 11 years ago, the sum of all human knowledge could be provided in just two books.

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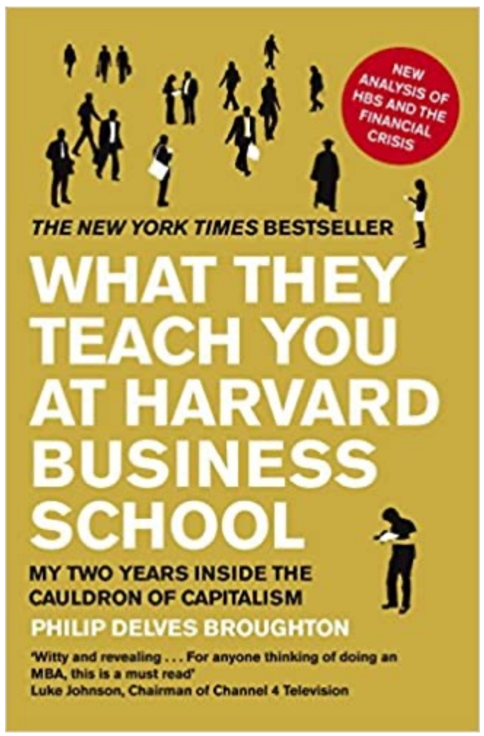

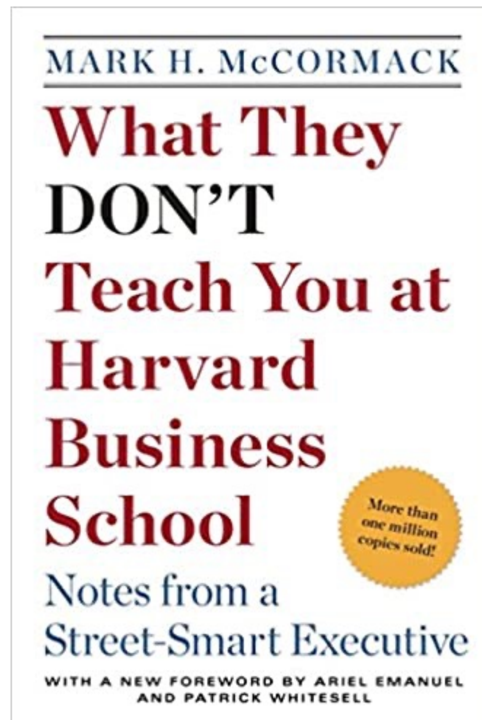

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Since x was arbitrary, we have shown, by definition, that $(A \cup B)^C = A^C \cap B^C$.

Proof technique:
To show C = D show
$x \in C \to x \in D$ and
$x \in D \to x \in C$

De Morgan's Laws

Formally, prove $\forall x\,(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

1. Let x be arbitrary
   - 2.1. $x \in (A \cup B)^C$ — Assumption
   - ...
   - 2.3. $x \in A^C \cap B^C$
2. $x \in (A \cup B)^C \to x \in A^C \cap B^C$ — Direct Proof
   - 3.1. $x \in A^C \cap B^C$ — Assumption
   - ...
   - 3.3. $x \in (A \cup B)^C$
3. $x \in A^C \cap B^C \to x \in (A \cup B)^C$ — Direct Proof
4. $(x \in (A \cup B)^C \to x \in A^C \cap B^C) \wedge (x \in A^C \cap B^C \to x \in (A \cup B)^C)$ — Intro $\wedge$: 2, 3
5. $x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C$ — Biconditional: 4
6. $\forall x\,(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$ — Intro $\forall$: 1-5

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$.

…

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$.

…

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, we have $x \in A^C \cap B^C$.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, $x \in A^C$ and $x \in B^C$, so we we have $x \in A^C \cap B^C$ by the definition of intersection.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall$x $(x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$.

…

Thus, $\neg(x \in A)$ and $\neg(x \in B)$, so $x \in A^C$ and $x \in B^C$ by the definition of compliment, and we can see that $x \in A^C \cap B^C$ by the definition of intersection.

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall \mathrm{x}\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$. Then, by the definition of complement, we have $\neg(x \in A \cup B)$. The latter says, by the definition of union, that $\neg(x \in A \vee x \in B)$, or equivalently $\neg(x \in A) \wedge \neg(x \in B)$ by De Morgan's law. Thus, we have $x \in A^C$ and $x \in B^C$ by the definition of compliment, and we can see that $x \in A^C \cap B^C$ by the definition of intersection.

Proof technique:
To show C = D show
$x \in \mathrm{C} \rightarrow x \in \mathrm{D}$ and
$x \in \mathrm{D} \rightarrow x \in \mathrm{C}$

De Morgan's Laws

Prove that $(A \cup B)^C = A^C \cap B^C$

Formally, prove $\forall x\ (x \in (A \cup B)^C \leftrightarrow x \in A^C \cap B^C)$

Proof: Let x be an arbitrary object.

Suppose $x \in (A \cup B)^C$.... Then, $x \in A^C \cap B^C$.

Suppose $x \in A^C \cap B^C$. Then, by the definition of intersection, we have $x \in A^C$ and $x \in B^C$. That is, we have $\neg(x \in A) \wedge \neg(x \in B)$, which is equivalent to $\neg(x \in A \vee x \in B)$ by De Morgan's law. The last is equivalent to $\neg(x \in A \cup B)$, by the definition of union, so we have shown $x \in (A \cup B)^C$, by the definition of complement.