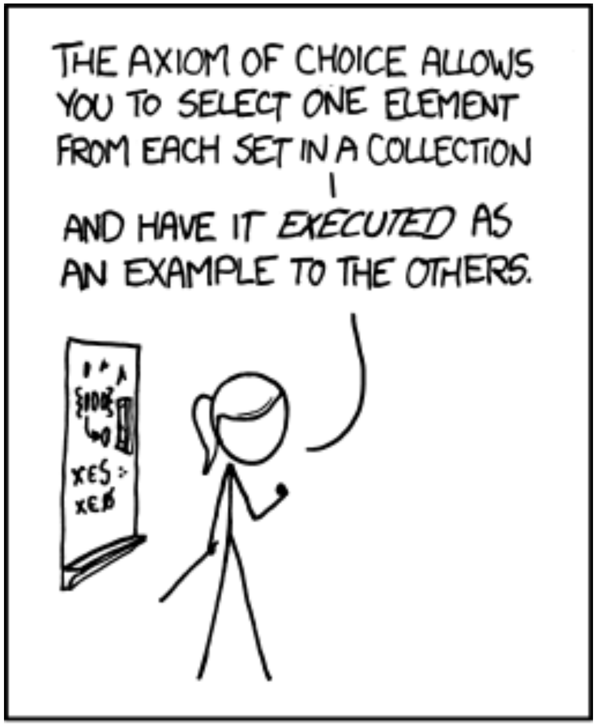
CSE 311: Foundations of Computing

Lecture 9: English Proofs & Proof Strategies



MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

Last class: Inference Rules for Quantifiers

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore \quad P(a) \textbf{ for any } a}$$

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \textbf{ for some } \textbf{\textit{special}}{**}\ c}$$

** c is a NEW name.

Intro ∀

$$\frac{\text{“}\textbf{Let } a \textbf{ be arbitrary}*\text{”} \ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

* in the domain of P.

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) := \exists y\ (x = 2 \cdot y)$
$\text{Odd}(x) := \exists y\ (x = 2 \cdot y + 1)$

Prove "There is an even number"

Formally: prove $\exists x\ \text{Even}(x)$

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) := \exists y\ (x = 2 \cdot y)$
$\text{Odd}(x) := \exists y\ (x = 2 \cdot y + 1)$

Prove "There is an even number"

Formally: prove $\exists x\ \text{Even}(x)$

1.	$\mathbf{2 = 2 \cdot 1}$	**Algebra**
2.	$\exists y\ (\mathbf{2} = \mathbf{2} \cdot y)$	**Intro** $\exists$**: 1**
3.	$\text{Even}(\mathbf{2})$	**Definition of** Even**: 2**
4.	$\exists x\ \text{Even}(x)$	**Intro** $\exists$**: 3**

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) := $\exists$y (x = 2·y) Odd(x) := $\exists$y (x = 2·y + 1) Prime(x) := "x > 1 and x≠a·b for all integers a, b with 1<a<x"

Prove "There is an even prime number"

Formally: prove $\exists x\,(\mathrm{Even}(x) \wedge \mathrm{Prime}(x))$

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) := $\exists y\ (x = 2 \cdot y)$
Odd(x) := $\exists y\ (x = 2 \cdot y + 1)$
Prime(x) := "x > 1 and $x \neq a \cdot b$ for all integers a, b with $1 < a < x$"

Prove "There is an even prime number"

Formally: prove $\exists x\ (\text{Even}(x) \wedge \text{Prime}(x))$

1.	$\mathbf{2 = 2 \cdot 1}$	**Algebra**
2.	$\exists y\ (\mathbf{2} = \mathbf{2} \cdot y)$	**Intro** $\exists$**: 1**
3.	Even(**2**)	**Def of** Even**: 3**
4.	Prime(**2**)*	**Property of integers**
5.	Even(**2**) $\wedge$ Prime(**2**)	**Intro** $\wedge$**: 2, 4**
6.	$\exists x\ (\text{Even}(x) \wedge \text{Prime}(x))$	**Intro** $\exists$**: 5**

*** Later we will further break down** "Prime" **using quantifiers to prove statements like this**

Inference Rules for Quantifiers: First look

Intro ∃
$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀
$$\frac{\forall x\, P(x)}{\therefore \quad P(a) \;\; \textbf{(for any } a)}$$

Elim ∃
$$\frac{\exists x\, P(x)}{\therefore\; P(c) \textbf{ for some } \textbf{\textit{special}}{**}\; c}$$

Intro ∀
$$\frac{\text{“}\textbf{Let } a \textbf{ be arbitrary*}\text{”}\ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

** By special, we mean that c is a name for a value where P(c) is true. We can’t use anything else about that value, so c has to be a NEW name!

* in the domain of P

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	"Let a be arbitrary*"...P(a)	Elim ∃	∃x P(x)
	∴ ∀x P(x)		∴ P(c) for some ***special**** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

3. $\forall x\,(\text{Even}(x)\rightarrow\text{Even}(x^2))$?

Even and Odd

Even(x) := $\exists$y (x=2y)
Odd(x) := $\exists$y (x=2y+1)
Domain: Integers

Intro $\forall$	Elim $\exists$
"**Let** a **be arbitrary***"...P(a) $\therefore$ $\forall$x P(x)	$\exists$x P(x) $\therefore$ P(c) **for some *special*** ** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2. Even(**a**)$\rightarrow$Even(**a**^{2}) ?
3. $\forall x\,(\text{Even}(x)\rightarrow\text{Even}(x^2))$ Intro $\forall$: 1,2

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

1. Let a be an arbitrary integer
2.1 Even(a) Assumption
2.6 Even(a²)
2. Even(a)→Even(a²) Direct proof rule
3. ∀x (Even(x)→Even(x²)) Intro ∀: 1,2

1. Let a be an arbitrary integer
2.1 Even(a) Assumption
2.6 Even(a²)
2. Even(a)→Even(a²) Direct proof rule
3. ∀x (Even(x)→Even(x²)) Intro ∀: 1,2

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

    2.1 $\text{Even}(a)$ — Assumption

    2.6 $\text{Even}(a^2)$ — ?

2. $\text{Even}(a) \rightarrow \text{Even}(a^2)$ — Direct proof
3. $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Even and Odd

Intro ∀	"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	Elim ∃	∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) — Assumption
   - 2.2 ∃y (**a** = 2y) — Definition of Even
   - **2.5** ∃y ($\mathbf{a}^2$ = 2y) — ?
   - 2.6 Even($\mathbf{a}^2$) — Definition of Even
2. Even(**a**)→Even($\mathbf{a}^2$) — Direct Proof
3. $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	Elim ∃
$\dfrac{\text{“Let a be arbitrary*”...P(a)}}{\therefore\ \forall x\, P(x)}$	$\dfrac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some special** } c}$

Prove: “The square of any even number is even.”

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) — Assumption
   - 2.2 $\exists y\,(\mathbf{a} = 2y)$ — Definition of Even
   - 2.5 $\exists y\,(\mathbf{a}^2 = 2y)$ — Intro ∃: ?
   - 2.6 Even($\mathbf{a}^2$) — Definition of Even
2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof
3. $\forall x\,(\text{Even}(x)\rightarrow\text{Even}(x^2))$ — Intro ∀: 1,2

Need $\mathbf{a}^2 = 2\mathbf{c}$ for some **c**

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Even and Odd

Intro ∀ $\dfrac{\text{“Let a be arbitrary*”...P(a)}}{\therefore \quad \forall x\, P(x)}$

1. Let a be an arbitrary integer
 2.1 Even(a) Assumption
 2.2 ∃y (a = 2y) Definition of Even
 2.3 a = 2b Elim ∃: b
 2.5 ∃y (a² = 2y) Intro ∃:
 2.6 Even(a²) Definition of Even
2. Even(a)→Even(a²) Direct proof
3. ∀x (Even(x)→Even(x²)) Intro ∀: 1,2

Prove: “The square of any even number is even.”

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1.	Let a be an arbitrary integer	
2.1	Even(a)	Assumption
2.2	$\exists y\,(a = 2y)$	Definition of Even
2.3	$a = 2b$	Elim ∃: b
2.5	$\exists y\,(a^2 = 2y)$	Intro ∃: ?
2.6	Even(a^2)	Definition of Even
2.	Even(a)→Even(a^2)	Direct proof
3.	$\forall x\,(\text{Even}(x)\rightarrow\text{Even}(x^2))$	Intro ∀: 1,2

Need $a^2 = 2c$ for some c

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	Elim ∃
"**Let** a **be arbitrary***"…P(a) ∴ ∀x P(x)	∃x P(x) ∴ P(c) **for some *special***** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	$\exists y\,(\mathbf{a} = 2y)$	Definition of Even
2.3	$\mathbf{a} = 2\mathbf{b}$	Elim ∃: **b**
2.4	$\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$	Algebra
2.5	$\exists y\,(\mathbf{a}^2 = 2y)$	Intro ∃
2.6	Even($\mathbf{a}^2$)	Definition of Even

Used $\mathbf{a}^2 = 2c$ for $c=2\mathbf{b}^2$

2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$ — Direct Proof
3. $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

These rules need some caveats…

There are extra conditions on using these rules:

Intro ∀ "Let a be arbitrary*"…P(a)

∴ ∀x P(x)

* in the domain of P. No other name in P depends on a

Elim ∃ ∃x P(x)

∴ P(c) for some *special*** c

** c is a NEW name.
List all dependencies for c.

Without those rules, it is possible to infer claims that are false

Formal Proofs

- In principle, formal proofs are the standard for what it means to be "proven" in mathematics
  - almost all math (and theory CS) done in Predicate Logic
- But they are tedious and impractical
  - e.g., applications of commutativity and associativity
  - Russell & Whitehead's formal proof that 1+1 = 2 is *several hundred pages* long

    we allowed ourselves to cite "Arithmetic", "Algebra", etc.
- Similar situation exists in programming…

Programming

```
a := ADD(i, 1)
b := MOD(a, n)
c := ADD(arr, b)
d := LOAD(c)
e := ADD(arr, i)
STORE(e, d)
```

```
arr[i] = arr[(i+1) % n];
```

Assembly Language

High-level Language

Programming vs Proofs

`a := ADD(i, 1)`	**Given**
`b := MOD(a, n)`	**Given**
`c := ADD(arr, b)`	**Elim ∧: 1**
`d := LOAD(c)`	**Double Negation: 4**
`e := ADD(arr, i)`	**Elim ∨: 3, 5**
`STORE(e, d)`	**Modus Ponens: 2, 6**

Assembly Language for Programs

Assembly Language for Proofs

Proofs

Given

Given

$\wedge$ Elim: 1

Double Negation: 4

$\vee$ Elim: 3, 5

MP: 2, 6

what is the "Java" for proofs?

Assembly Language for Proofs

High-level Language for Proofs

Proofs

Given

Given

∧ Elim: 1

Double Negation: 4

∨ Elim: 3, 5

MP: 2, 6

English?

Assembly Language for Proofs

High-level Language for Proofs

Proofs

Given

Given

∧ Elim: 1

Double Negation: 4

∨ Elim: 3, 5

MP: 2, 6

Math English

Assembly Language for Proofs

High-level Language for Proofs

Proofs

- Formal proofs follow simple well-defined rules and should be easy for a machine to check
  - as assembly language is easy for a machine to execute

- English proofs correspond to those rules but are designed to be easier for humans to read
  - also easy to check with practice

    (almost all actual math and theory CS is done this way)
  - English proof is correct if the <u>reader</u> believes they could translate it into a formal proof

    (the reader is the “compiler” for English proofs)

Last class: Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove: "The square of every even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	$\exists y\ (\mathbf{a} = 2y)$	Definition of Even
2.3	$\mathbf{a} = 2\mathbf{b}$	Elim $\exists$
2.4	$\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$	Algebra
2.5	$\exists y\ (\mathbf{a}^2 = 2y)$	Intro $\exists$
2.6	Even($\mathbf{a}^2$)	Definition of Even

2. Even(**a**)$\rightarrow$Even($\mathbf{a}^2$) — Direct Proof
3. $\forall x\,(\text{Even}(x)\rightarrow\text{Even}(x^2))$ — Intro $\forall$

English Proof: Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Let a be an arbitrary integer.	1.	Let a be an arbitrary integer	
Suppose a is even.	2.1	$\text{Even}(a)$	Assumption
Then, by definition, $a = 2b$ for some integer b.	2.2	$\exists y\ (a = 2y)$	Definition
	2.3	$a = 2b$	Elim $\exists$
Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$.	2.4	$a^2 = 4b^2 = 2(2b^2)$	Algebra
So a^2 is, by definition, even.	2.5	$\exists y\ (a^2 = 2y)$	Intro $\exists$
	2.6	$\text{Even}(a^2)$	Definition
Since a was arbitrary, we have shown that the square of every even number is even.	2.	$\text{Even}(a) \rightarrow \text{Even}(a^2)$	Direct Proof
	3.	$\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$	Intro $\forall$

English Proof: Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Proof: Let **a** be an arbitrary integer.

Suppose **a** is even. Then, by definition, **a** = 2**b** for some integer b. Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$. So a^2 is, by definition, is even.

Since **a** was arbitrary, we have shown that the square of every even number is even. ■

English Proof: Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Proof: Let **a** be an arbitrary **even** integer.

Then, by definition, $\mathbf{a} = 2\mathbf{b}$ for some integer b. Squaring both sides, we get $\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$. So $\mathbf{a}^2$ is, by definition, is even.

Since **a** was arbitrary, we have shown that the square of every even number is even. ■

$$\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$$

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\mathrm{Odd}(x) \wedge \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall$x $\forall$y ((Odd(x) ∧ Odd(y))→Even(x+y))

Let x and y be arbitrary integers.

1. Let **x** be an arbitrary integer
2. Let **y** be an arbitrary integer

Since x and y were arbitrary, the sum of any odd integers is even.

3. (Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**)
4. $\forall$x $\forall$y ((Odd(**x**) ∧ Odd(**y**)) → Even(x+y)) Intro $\forall$

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\mathrm{Odd}(x) \land \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$

Let x and y be arbitrary integers.

1. **Let x be an arbitrary integer**
2. **Let y be an arbitrary integer**

Suppose that both are odd.

3.1 Odd(**x**) ∧ Odd(**y**) Assumption

so x+y is even.

3.9 Even(**x+y**)

Since x and y were arbitrary, the sum of any odd integers is even.

3. (Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**) DPR
4. ∀x ∀y ((Odd(**x**) ∧ Odd(**y**)) → Even(x+y)) Intro ∀

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\text{Odd}(x) \wedge \text{Odd}(y)) \rightarrow \text{Even}(x+y))$

Let x and y be arbitrary integers.

1. Let **x** be an arbitrary integer
2. Let **y** be an arbitrary integer

Suppose that both are odd.

3.1 Odd(**x**) ∧ Odd(**y**) Assumption
3.2 Odd(**x**) Elim ∧: 2.1
3.3 Odd(**y**) Elim ∧: 2.1

so x+y is even.

3.9 Even(**x+y**)

Since x and y were arbitrary, the sum of any odd integers is even.

3. (Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**) DPR
4. ∀x ∀y ((Odd(**x**) ∧ Odd(**y**)) → Even(x+y)) Intro ∀

English Proof: Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The sum of two odd numbers is even."

English		Formal	Justification
Let x and y be arbitrary integers.	1.	Let **x** be an arbitrary integer	
	2.	Let **y** be an arbitrary integer	
	3.1	Odd(**x**) ∧ Odd(**y**)	Assumption
Suppose that both are odd.	3.2	Odd(**x**)	Elim ∧: 2.1
	3.3	Odd(**y**)	Elim ∧: 2.1
Then, we have x = 2a+1 for some integer a and y = 2b+1 for some integer b.	3.4	∃**z** (**x** = 2**z**+1)	Def of Odd: 2.2
	3.5	**x** = 2**a**+1	Elim ∃: 2.4
	3.6	∃**z** (**y** = 2**z**+1)	Def of Odd: 2.3
	3.7	**y** = 2**b**+1	Elim ∃: 2.5
	3.9	∃z (**x+y** = 2z)	Intro ∃: 2.4
so x+y is, by definition, even.	3.10	Even(**x+y**)	Def of Even
Since x and y were arbitrary, the sum of any odd integers is even.	3.	(Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**)	DPR
	4.	∀x ∀y ((Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**))	Intro ∀

English Proof: Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The sum of two odd numbers is even."

English	Formal
Let x and y be arbitrary integers.	1. **Let x be an arbitrary integer** 2. **Let y be an arbitrary integer**
Suppose that both are odd.	3.1 Odd(x) ∧ Odd(y) — Assumption 3.2 Odd(x) — Elim ∧: 2.1 3.3 Odd(y) — Elim ∧: 2.1
Then, we have x = 2a+1 for some integer a and y = 2b+1 for some integer b.	3.4 ∃z (x = 2z+1) — Def of Odd: 2.2 3.5 x = 2a+1 — Elim ∃: 2.4 3.6 ∃z (y = 2z+1) — Def of Odd: 2.3 3.7 y = 2b+1 — Elim ∃: 2.5
Their sum is x+y = ... = 2(a+b+1)	3.8 x+y = 2(a+b+1) — Algebra
so x+y is, by definition, even.	3.9 ∃z (x+y = 2z) — Intro ∃: 2.4 3.10 Even(x+y) — Def of Even
Since x and y were arbitrary, the sum of any odd integers is even.	3. (Odd(x) ∧ Odd(y)) → Even(x+y) — DPR 4. ∀x ∀y ((Odd(x) ∧ Odd(y)) → Even(x+y)) — Intro ∀

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Proof: Let x and y be arbitrary integers.

Suppose that both are odd. Then, we have x = 2a+1 for some integer a and y = 2b+1 for some integer b. Their sum is x+y = (2a+1) + (2b+1) = 2a+2b+2 = 2(a+b+1), so x+y is, by definition, even.

Since x and y were arbitrary, the sum of any two odd integers is even. ■

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Proof: Let x and y be arbitrary **odd** integers.

Then, x = 2a+1 for some integer a and y = 2b+1 for some integer b. Their sum is x+y = (2a+1) + (2b+1) = 2a+2b+2 = 2(a+b+1), so x+y is, by definition, even.

Since x and y were arbitrary, the sum of any two odd integers is even.

■

$\forall x\ \forall y\ ((\mathrm{Odd}(x) \wedge \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$

Rational Numbers

Domain of Discourse
Real Numbers

- A real number x is *rational* iff there exist integers a and b with $b \neq 0$ such that $x = a/b$.

$$\text{Rational}(x) := \exists a\, \exists b\, (((\text{Integer}(a) \wedge \text{Integer}(b)) \wedge (x = a/b)) \wedge b \neq 0)$$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

Formally, prove $\forall x\ \forall y\ ((\text{Rational}(x) \land \text{Rational}(y)) \rightarrow \text{Rational}(xy))$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

Proof: Let x and y be arbitrary rationals.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

Proof: Let x and y be arbitrary rationals.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

By definition, then, xy is rational.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

Proof: Let x and y be arbitrary rationals.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

Multiplying, we get that xy = (a/b)(c/d) = (ac)/(bd). Since b and d are both non-zero, so is bd. Furthermore, ac and bd are integers. By definition, then, xy is rational.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

OR "If x and y are rational, then xy is rational."

Recall that unquantified variables (not constants) are implicitly for-all quantified.

$\forall x\ \forall y\ ((\text{Rational}(x) \land \text{Rational}(y)) \rightarrow \text{Rational}(xy))$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Proof: ~~Let x and y be arbitrary rationals.~~

Suppose x and y are rational.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

Multiplying, we get that xy = (a/b)(c/d) = (ac)/(bd). Since b and d are both non-zero, so is bd. Furthermore, ac and bd are integers. By definition, then, xy is rational.

~~Since x and y were arbitrary, we have shown that the product of any two rationals is rational.~~ ■

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Suppose x and y are rational.

1.1 $\text{Rational}(x) \wedge \text{Rational}(y)$ **Assumption**

Then, x = a/b for some integers a, b, where b≠0 and y = c/d for some integers c,d, where d≠0.

1.4 $\exists p\, \exists q\, ((x = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge (q \neq 0))$ **Def Rational: 1.2**

1.5 $(x = a/b) \wedge \text{Integer}(a) \wedge \text{Integer}(b) \wedge (b \neq 0)$ **Elim ∃: 1.4**

1.6 $\exists p\, \exists q\, ((x = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge (q \neq 0))$ **Def Rational: 1.3**

1.7 $(y = c/d) \wedge \text{Integer}(c) \wedge \text{Integer}(d) \wedge (d \neq 0)$ **Elim ∃: 1.4**

...

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: “If x and y are rational, then xy is rational.”

Suppose x and y are rational.

1.1 $\text{Rational}(x) \land \text{Rational}(y)$ **Assumption**

??

Then, x = a/b for some integers a, b, where b≠0 and y = c/d for some integers c,d, where d≠0.

...

1.4 $\exists p\, \exists q\, ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$ **Def Rational: 1.2**

1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$ **Elim ∃: 1.4**

1.6 $\exists p\, \exists q\, ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$ **Def Rational: 1.3**

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$ **Elim ∃: 1.4**

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Suppose x and y are rational.

Then, x = a/b for some integers a, b, where b≠0 and y = c/d for some integers c,d, where d≠0.

...

1.1 $\text{Rational}(x) \land \text{Rational}(y)$ **Assumption**

1.2 $\text{Rational}(x)$ **Elim ∧: 1.1**

1.3 $\text{Rational}(y)$ **Elim ∧: 1.1**

1.4 $\exists p\ \exists q\ ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$
Def Rational: 1.2

1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$
Elim ∃: 1.4

1.6 $\exists p\ \exists q\ ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$
Def Rational: 1.3

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$
Elim ∃: 1.4

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

...

1.5 $(x = a/b) \wedge \text{Integer}(a) \wedge \text{Integer}(b) \wedge (b \neq 0)$

...

1.7 $(y = c/d) \wedge \text{Integer}(c) \wedge \text{Integer}(d) \wedge (d \neq 0)$

Multiplying, we get xy = (ac)/(bd).

1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$

Algebra

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

...

1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$

...

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$

??

Multiplying, we get xy = (ac)/(bd).

1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$ **Algebra**

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Multiplying, we get xy = (ac)/(bd).

…

1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$

…

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$

1.8 $x = a/b$ **Elim ∧: 1.5**

1.9 $y = c/d$ **Elim ∧: 1.7**

1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$

Algebra

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

...

1.5 $(x = a/b) \wedge \text{Integer}(a) \wedge \text{Integer}(b) \wedge (b \neq 0)$

...

1.7 $(y = c/d) \wedge \text{Integer}(c) \wedge \text{Integer}(d) \wedge (d \neq 0)$

...

	1.11	$b \neq 0$	**Elim $\wedge$: 1.5***
	1.12	$d \neq 0$	**Elim $\wedge$: 1.7**
Since b and d are non-zero, so is bd.	**1.13**	$bd \neq 0$	**Prop of Integer Mult**

*** Oops, I skipped steps here...**

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

...

1.5 $(x = a/b) \land (\text{Integer}(a) \land (\text{Integer}(b) \land (b \neq 0)))$

...

1.7 $(y = c/d) \land (\text{Integer}(c) \land (\text{Integer}(d) \land (d \neq 0)))$

...

1.11 $\text{Integer}(a) \land \left(\text{Integer}(b) \land (b \neq 0)\right)$ **Elim ∧: 1.5**

1.12 $\text{Integer}(b) \land (b \neq 0)$ **Elim ∧: 1.11**

1.13 $b \neq 0$ **Elim ∧: 1.12**

We left out the parentheses...

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\mathrm{Integer}(a) \land \mathrm{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

...

1.5 $(x = a/b) \land \mathrm{Integer}(a) \land \mathrm{Integer}(b) \land (b \neq 0)$

...

1.7 $(y = c/d) \land \mathrm{Integer}(c) \land \mathrm{Integer}(d) \land (d \neq 0)$

...

1.13 $b \neq 0$ — **Elim ∧: 1.5**

...

1.16 $d \neq 0$ — **Elim ∧: 1.7**

Since b and d are non-zero, so is bd. | **1.17** $bd \neq 0$ — **Prop of Integer Mult**

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

...

1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$

...

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$

...

1.19 $\text{Integer}(a)$ **Elim ∧: 1.5***

...

1.22 $\text{Integer}(b)$ **Elim ∧: 1.5***

...

1.24 $\text{Integer}(c)$ **Elim ∧: 1.7***

...

1.27 $\text{Integer}(d)$ **Elim ∧: 1.7***

1.28 $\text{Integer}(ac)$ **Prop of Integer Mult**

1.29 $\text{Integer}(bd)$ **Prop of Integer Mult**

Furthermore, ac and bd are integers.

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

...

1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$

...

1.17 $bd \neq 0$ **Prop of Integer Mult**

...

1.28 $\text{Integer}(ac)$ **Prop of Integer Mult**

1.29 $\text{Integer}(bd)$ **Prop of Integer Mult**

1.30 $\text{Integer}(bd) \land (bd \neq 0)$ **Intro ∧: 1.29, 1.17**

1.31 $\text{Integer}(ac) \land \text{Integer}(bd) \land (bd \neq 0)$

Intro ∧: 1.28, 1.30

1.32 $(xy = (a/b)/(c/d)) \land \text{Integer}(ac) \land$
$\text{Integer}(bd) \land (bd \neq 0)$ **Intro ∧: 1.10, 1.31**

1.33 $\exists p\, \exists q\, ((xy = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$

Intro ∃: 1.32

1.34 $\text{Rational}(xy)$ **Def of Rational: 1.3**

By definition, then, xy is rational.

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Suppose x and y are rational.	**1.1** $\text{Rational}(x) \land \text{Rational}(y)$	**Assumption**
	...	
	1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$	
	...	
	1.17 $bd \neq 0$	**Prop of Integer Mult**
	...	
Furthermore, ac and bd are integers.	**1.28** $\text{Integer}(ac)$	**Prop of Integer Mult**
	1.29 $\text{Integer}(bd)$	**Prop of Integer Mult**
	...	
By definition, then, xy is rational.	**1.34** $\text{Rational}(xy)$	**Def of Rational: 1.32**

And finally...

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Suppose x and y are rational.	**1.1** $\text{Rational}(x) \wedge \text{Rational}(y)$	**Assumption**
	...	
	1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$	
	...	
	1.17 $bd \neq 0$	**Prop of Integer Mult**
	...	
Furthermore, ac and bd are integers.	**1.28** $\text{Integer}(ac)$	**Prop of Integer Mult**
	1.29 $\text{Integer}(bd)$	**Prop of Integer Mult**
	...	
By definition, then, xy is rational.	**1.34** $\text{Rational}(xy)$	**Def of Rational: 1.32**

1. $\text{Rational}(x) \wedge \text{Rational}(y) \rightarrow \text{Rational}(xy)$

Direct Proof

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Proof: Suppose x and y are rational.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

Multiplying, we get that xy = (ac)/(bd). Since b and d are both non-zero, so is bd. Furthermore, ac and bd are integers. By definition, then, xy is rational. ■

vs more than 35 lines of formal proof

English Proofs

- High-level language let us work more quickly
  - should not be necessary to spill out every detail
  - <u>reader</u> checks that the writer is not skipping too much
  - examples so far

    skipping Intro ∧ and Elim ∧

    not stating existence claims (immediately apply Elim ∃ to name the object)

    not stating that the implication has been proven ("Suppose X... Thus, Y." says it already)
  - (list will grow over time)

- English proof is correct if the <u>reader</u> believes they could translate it into a formal proof
  - the reader is the "compiler" for English proofs