Lecture 8: Predicate Logic Proofs

... let's assume there exists some function $F(a, b, c...)$ which produces the correct answer—hang on.

This is going to be one of those weird, dark-magic proofs, isn't it? I can tell.

What? No, no; it's a perfectly sensible chain of reasoning.

All right...

Now, let's assume the correct answer will eventually be written on this board at the coordinates $(x, y)$. If we—

I knew it!
Two inference rules per binary connective, one to eliminate it and one to introduce it

- Elimination of conjunction: \( A \land B \vdash A, B \)
- Introduction of conjunction: \( A ; B \vdash A \land B \)
- Elimination of disjunction: \( A \lor B ; \neg A \vdash B \)
- Introduction of disjunction: \( A \vdash A \lor B, B \lor A \)
- Modus Ponens: \( A ; A \rightarrow B \vdash B \)
- Direct Proof: \( A \Rightarrow B \vdash A \rightarrow B \)
Proofs

Show that $r$ follows from $p$, $p \rightarrow q$ and $(p \land q) \rightarrow r$

How To Start:
We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

\[
A ; A \rightarrow B \\
\therefore B
\]

\[
A \land B \\
\therefore A, B
\]

\[
A ; B \\
\therefore A \land B
\]
Proofs

Show that \( r \) follows from \( p, p \rightarrow q \) and \( (p \land q) \rightarrow r \)

1. \( p \) \hspace{2cm} \text{Given} \hspace{2cm} \text{A ; A } \rightarrow \text{ B} \hspace{2cm} \therefore \ B

2. \( p \rightarrow q \) \hspace{2cm} \text{Given} \hspace{2cm} \text{A } \land \text{ B} \hspace{2cm} \therefore \ A, \ B

3. \( p \land q \rightarrow r \) \hspace{2cm} \text{Given} \hspace{2cm} \text{A ; B} \hspace{2cm} \therefore \ A \land B

9. \( r \) \hspace{2cm} \text{??} \hspace{2cm} \therefore \ A \land B
Proofs

Show that \( r \) follows from \( p, p \to q, \) and \( p \land q \to r \)

1. \( p \) \hspace{2cm} \text{Given}
2. \( p \to q \) \hspace{2cm} \text{Given}
3. \( q \) \hspace{2cm} \text{MP: 1, 2}
4. \( p \land q \) \hspace{2cm} \text{Intro } \land: 1, 3
5. \( p \land q \to r \) \hspace{2cm} \text{Given}
6. \( r \) \hspace{2cm} \text{MP: 4, 5}

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that’s great!
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \)  
   Given

2. \( q \rightarrow \neg r \)  
   Given

3. \( \neg s \lor q \)  
   Given

20. \( \neg r \)  
   Idea: Work backwards!
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \)  
   Given

2. \( q \rightarrow \neg r \)  
   Given

3. \( \neg s \lor q \)  
   Given

Idea: Work backwards!
We want to eventually get \( \neg r \). How?
- We can use \( q \rightarrow \neg r \) to get there.
- The justification between 2 and 20 looks like “elim \( \rightarrow \)” which is MP.

20. \( \neg r \)  
   MP: 2, ?
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

**Idea: Work backwards!**

We want to eventually get \( \neg r \). How?

- Now, we have a new “hole”
- We need to prove \( q \)...
  - Notice that at this point, if we prove \( q \), we’ve proven \( \neg r \)...

19. \( q \)  
20. \( \neg r \) MP: 2, 19
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

This looks like or-elimination.

19. \( q \) 
20. \( \neg r \) MP: 2, 19
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

18. \( \neg\neg s \) \( \neg \neg s \) doesn’t show up in the givens but \( s \) does and we can use equivalences

19. \( q \lor \) Elim: 3, 18
20. \( \neg r \) MP: 2, 19
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

17. \( s \) 
18. \( \neg \neg s \) Double Negation: 17
19. \( q \) \lor Elim: 3, 18
20. \( \neg r \) MP: 2, 19
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given
17. \( s \) \land Elim: 1
18. \( \neg \neg s \) Double Negation: 17
19. \( q \) \lor Elim: 3, 18
20. \( \neg r \) MP: 2, 19

No holes left! We just need to clean up a bit.
Proofs

Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given
4. \( s \) \( \land \) Elim: 1
5. \( \neg \neg s \) Double Negation: 4
6. \( q \) \( \lor \) Elim: 3, 5
7. \( \neg r \) MP: 2, 6
Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.
  e.g. \((p \rightarrow r) \lor q \equiv (\neg p \lor r) \lor q\)

- Inference rules only can be applied to whole formulas (not correct otherwise).
  e.g. 1. \(p \rightarrow r\) given
       2. \((p \lor q) \rightarrow r\) intro \(\lor\) from 1.

Does not follow! e.g. \(p=F, q=T, r=F\)
Last class: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

\[ \begin{align*}
\text{Elim } \land & : \quad A \land B \\ & \quad : A, B
\end{align*} \]

\[ \begin{align*}
\text{Intro } \land & : \quad A ; B \\ & \quad : A \land B
\end{align*} \]

\[ \begin{align*}
\text{Elim } \lor & : \quad A \lor B ; \neg A \\ & \quad : B
\end{align*} \]

\[ \begin{align*}
\text{Intro } \lor & : \quad A \\ & \quad : A \lor B, B \lor A
\end{align*} \]

\[ \begin{align*}
\text{Modus Ponens} & : \quad A ; A \to B \\ & \quad : B
\end{align*} \]

\[ \begin{align*}
\text{Direct Proof} & : \quad A \Rightarrow B \\ & \quad : A \to B
\end{align*} \]

Not like other rules
Last class: New Perspective

Rather than comparing \( A \) and \( B \) as columns, zooming in on just the rows where \( A \) is true:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( q )</td>
<td>( A )</td>
<td>( B )</td>
</tr>
<tr>
<td>T</td>
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<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>

Given that \( A \) is true, we see that \( B \) is also true.

\[ A \Rightarrow B \]
Last class: New Perspective

Rather than comparing $A$ and $B$ as columns, zooming in on just the rows where $B$ is true:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$A$</th>
<th>$B$</th>
<th>$A \rightarrow B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
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</tr>
</tbody>
</table>

When we zoom out, what have we proven?

\[(A \rightarrow B) \equiv T\]
To Prove An Implication: \( A \rightarrow B \)

- We use the direct proof rule

\[
A \Rightarrow B \\
\therefore A \rightarrow B
\]

- The “pre-requisite” \( A \Rightarrow B \) for the direct proof rule is a proof that “Given \( A \), we can prove \( B \).”

- The direct proof rule:
  
  If you have such a proof then you can conclude that \( A \rightarrow B \) is true
Proofs using the direct proof rule

Show that $p \rightarrow r$ follows from $q$ and $(p \land q) \rightarrow r$

1. $q$ Given
2. $(p \land q) \rightarrow r$ Given

This is a proof of $p \rightarrow r$

3.1. $p$ Assumption
3.2.
3.3. $r$ ??

3. $p \rightarrow r$ Direct Proof

If we know $p$ is true...

Then, we’ve shown $r$ is true
Proofs using the direct proof rule

Show that $p \rightarrow r$ follows from $q$ and $(p \land q) \rightarrow r$

1. $q$ Given
2. $(p \land q) \rightarrow r$ Given
   
   3.1. $p$ Assumption
   3.2. $p \land q$ Intro $\land$: 1, 3.1
   3.3. $r$ MP: 2, 3.2

3. $p \rightarrow r$ Direct Proof
Example

Prove: \((p \land q) \rightarrow (p \lor q)\)

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens...
Example

Prove: \((p \land q) \rightarrow (p \lor q)\)

1.1. \(p \land q\) \hspace{1cm} \text{Assumption}

1.9. \(p \lor q\) \hspace{1cm} ??

1. \((p \land q) \rightarrow (p \lor q)\) \hspace{1cm} \text{Direct Proof}
Example

Prove: \((p \land q) \rightarrow (p \lor q)\)

1.1. \(p \land q\) Assumption

1.2. \(p\) Elim \(\land\): 1.1

1.3. \(p \lor q\) Intro \(\lor\): 1.2

1. \((p \land q) \rightarrow (p \lor q)\) Direct Proof
One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.
Example

Prove: \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)
Example

Prove: \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)

1.1. \((p \rightarrow q) \land (q \rightarrow r)\)  Assumption

1.? \(p \rightarrow r\)

1. \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)  Direct Proof
Example

Prove: \[ ((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r) \]

1.1. \((p \rightarrow q) \land (q \rightarrow r)\) Assumption

1.2. \(p \rightarrow q\) \(\land\) Elim: 1.1

1.3. \(q \rightarrow r\) \(\land\) Elim: 1.1

1.? \(p \rightarrow r\)

1. \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\) Direct Proof
Example

Prove: \((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)

1.1. \((p \rightarrow q) \land (q \rightarrow r)\) Assumption

1.2. \(p \rightarrow q\) \land Elim: 1.1

1.3. \(q \rightarrow r\) \land Elim: 1.1

1.4.1. \(p\) Assumption

1.4.? \(r\)

1.4. \(p \rightarrow r\) Direct Proof

1. \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\) Direct Proof
Example

Prove: \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)

1.1. \((p \rightarrow q) \land (q \rightarrow r)\) Assumption
1.2. \(p \rightarrow q\) \land\ Elim: 1.1
1.3. \(q \rightarrow r\) \land\ Elim: 1.1

1.4.1. \(p\) Assumption
1.4.2. \(q\) MP: 1.2, 1.4.1
1.4.3. \(r\) MP: 1.3, 1.4.2

1.4. \(p \rightarrow r\) Direct Proof

1. \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\) Direct Proof
By special, we mean that c is a name for a value where P(c) is true. We can’t use anything else about that value, so c has to be a NEW name!
My First Predicate Logic Proof

Prove \((\forall x \, P(x)) \rightarrow (\exists x \, P(x))\)

The main connective is implication so Direct Proof seems good.
My First Predicate Logic Proof

Prove $\forall x P(x) \rightarrow \exists x P(x)$

1. $\forall x P(x)$ Assumption

We need an $\exists$ we don’t have so “intro $\exists$” rule makes sense

1.5. $\exists x P(x)$

1. $\forall x P(x) \rightarrow \exists x P(x)$ Direct Proof
My First Predicate Logic Proof

Prove $\forall x P(x) \rightarrow \exists x P(x)$

1. $\forall x P(x)$ Assumption

1.5. $\exists x P(x)$ Intro \exists: ?

That requires $P(c)$ for some $c$.

We need an $\exists$ we don’t have so “intro $\exists$” rule makes sense.

1. $\forall x P(x) \rightarrow \exists x P(x)$ Direct Proof
My First Predicate Logic Proof

Prove $\forall x \, P(x) \rightarrow \exists x \, P(x)$

1. $\forall x \, P(x)$ Assumption

1.4. $P(5)$

1.5. $\exists x \, P(x)$ Intro $\exists$: 1.4

1. $\forall x \, P(x) \rightarrow \exists x \, P(x)$ Direct Proof
My First Predicate Logic Proof

Prove: \( \forall x \, P(x) \rightarrow \exists x \, P(x) \)

1. \( \forall x \, P(x) \)  
   Assumption

1.4. \( P(5) \)  
   Elim \( \forall \): 1.1

1.5. \( \exists x \, P(x) \)  
   Intro \( \exists \): 1.4

1. \( \forall x \, P(x) \rightarrow \exists x \, P(x) \)  
   Direct Proof
My First Predicate Logic Proof

Prove $\forall x \, P(x) \rightarrow \exists x \, P(x)$

1. $\forall x \, P(x)$ Assumption
2. $P(5)$ Elim $\forall$: 1.1
3. $\exists x \, P(x)$ Intro $\exists$: 1.2

1. $\forall x \, P(x) \rightarrow \exists x \, P(x)$ Direct Proof

Working forwards as well as backwards:
In applying “Intro $\exists$” rule we didn’t know what expression we might be able to prove $P(c)$ for, so we worked forwards to figure out what might work.
Predicate Logic Proofs

• Can use
  – Predicate logic inference rules
    whole formulas only
  – Predicate logic equivalences (De Morgan’s)
    even on subformulas
  – Propositional logic inference rules
    whole formulas only
  – Propositional logic equivalences
    even on subformulas
Predicate Logic Proofs with more content

• In propositional logic we could just write down other propositional logic statements as “givens”

• Here, we also want to be able to use domain knowledge so proofs are about something specific

• Example:

<table>
<thead>
<tr>
<th>Domain of Discourse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integers</td>
</tr>
</tbody>
</table>

• Given the basic properties of arithmetic on integers, define:

<table>
<thead>
<tr>
<th>Predicate Definitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even(x) := \exists y (x = 2 \cdot y)</td>
</tr>
<tr>
<td>Odd(x) := \exists y (x = 2 \cdot y + 1)</td>
</tr>
</tbody>
</table>