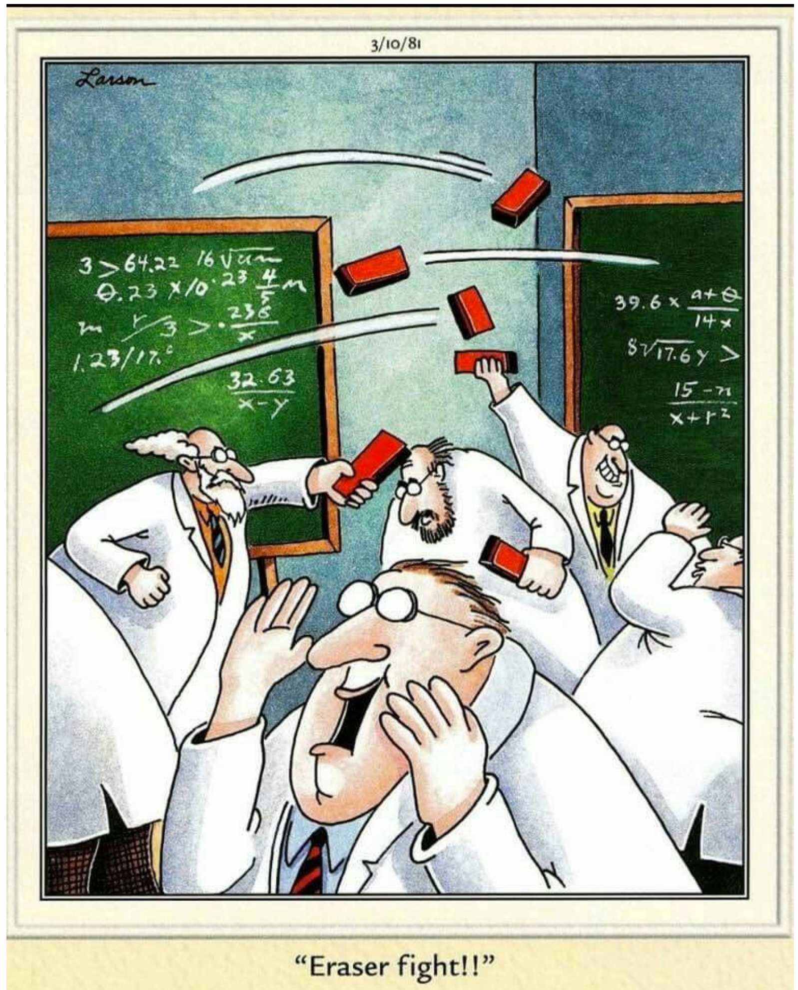
CSE 311: Foundations of Computing

Lecture 7: Logical Inference



“Eraser fight!!”

Last Class: Quantifiers

We use ***quantifiers*** to talk about collections of objects.

$\forall x\ P(x)$

$P(x)$ is true for every x in the domain

read as "for all x, P of x"

$\exists x\ P(x)$

There is an x in the domain for which $P(x)$ is true

read as "there exists x, P of x"

Statements with Quantifiers (Natural Translations)

Translations often (not always) sound more natural if we

1. Notice "domain restriction" patterns

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

Every prime number is either 2 or odd.

2. Avoid introducing *unnecessary* variable names

$\forall x\ \exists y\ \text{Greater}(y, x)$

For every positive integer, there is some larger positive integer.

3. Can sometimes drop "all" or "there is"

$\neg \exists x\ (\text{Even}(x) \wedge \text{Prime}(x) \wedge \text{Greater}(x, 2))$

No even prime is greater than 2.

More English Ambiguity

Implicit quantifiers in English are often **confusing**

<u>Three people that</u> are all friends can form a raiding party $\forall$

<u>Three people that</u> I know are all friends with Mark Zuckerberg $\exists$

Formal logic removes this ambiguity

- quantifiers can always be specified
- unquantified variables that are not known constants (e.g, π) are **implicitly** $\forall$–quantified

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= “x is a purple fruit”

(*) $\forall$x PurpleFruit(x) (“All fruits are purple”)

What is the negation of (*)?

(a) “there exists a purple fruit”

(b) “there exists a non-purple fruit”

(c) “all fruits are not purple”

Try your intuition! Which one seems right?

Negations of Quantifiers

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(b) "there exists a non-purple fruit"

(c) "all fruits are not purple"

Domain of Discourse
{plum, apple}

(*) PurpleFruit(plum) $\wedge$ PurpleFruit(apple)

(a) PurpleFruit(plum) $\vee$ PurpleFruit(apple)

(b) $\neg$ PurpleFruit(plum) $\vee$ $\neg$ PurpleFruit(apple)

(c) $\neg$ PurpleFruit(plum) $\wedge$ $\neg$ PurpleFruit(apple)

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

De Morgan's Laws for Quantifiers

$$\boxed{\begin{array}{c}\neg\forall x\, P(x) \equiv \exists x\, \neg P(x) \\ \neg \exists x\, P(x) \equiv \forall x\, \neg P(x)\end{array}}$$

These are **equivalent** but not **equal**

They have different English translations, e.g.:

There is no unicorn $\quad \neg \exists x\, \mathrm{Unicorn}(x)$

Every animal is not a unicorn $\quad \forall x\, \neg\, \mathrm{Unicorn}(x)$

De Morgan’s Laws for Quantifiers

$$\boxed{\begin{array}{c}\neg\forall x\, P(x) \equiv \exists x \neg P(x) \\ \neg\exists x\, P(x) \equiv \forall x \neg P(x)\end{array}}$$

“There is no integer at least as large as every other integer”

$$\begin{aligned} &\neg\exists x \forall y\ (x \geq y) \\ \equiv\ &\forall x \neg\forall y\ (x \geq y) \\ \equiv\ &\forall x\ \exists y \neg(x \geq y) \\ \equiv\ &\forall x\ \exists y\ (y > x) \end{aligned}$$

“For every integer, there is a larger integer”

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

"No even prime is greater than 2"

$$\neg \exists x\, (\mathrm{Even}(x) \wedge \mathrm{Prime}(x) \wedge \mathrm{Greater}(x, 2))$$
$$\equiv \forall x \neg(\mathrm{Even}(x) \wedge \mathrm{Prime}(x) \wedge \mathrm{Greater}(x, 2))$$
$$\equiv \forall x\, (\neg(\mathrm{Even}(x) \wedge \mathrm{Prime}(x)) \vee \neg\mathrm{Greater}(x, 2))$$
$$\equiv \forall x\, ((\mathrm{Even}(x) \wedge \mathrm{Prime}(x)) \rightarrow \neg\mathrm{Greater}(x, 2))$$
$$\equiv \forall x\, ((\mathrm{Even}(x) \wedge \mathrm{Prime}(x)) \rightarrow \mathrm{LessEq}(x, 2))$$

"Every even prime is less than or equal to 2."

De Morgan's Laws for Quantifiers

We just saw that

$$\neg \exists x\,(P(x) \wedge R(x)) \equiv \forall x\,(P(x) \rightarrow \neg R(x))$$

Can similarly show that

$$\neg \forall x\,(P(x) \rightarrow R(x)) \equiv \exists x\,(P(x) \wedge \neg R(x))$$

De Morgan's Laws respect domain restrictions!
(It leaves them in place and only negates the other parts.)

De Morgan's Laws for Quantifiers

$$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Remain true when domain restrictions are used:

$$\neg\exists x\, (P(x) \wedge R(x)) \equiv \forall x\, (P(x) \rightarrow \neg R(x))$$
$$\neg\forall x\, (P(x) \rightarrow R(x)) \equiv \exists x\, (P(x) \wedge \neg R(x))$$

Nested Quantifiers

- **Quantified variable names don't matter**

$$\forall x\, \exists y\, P(x, y) \equiv \forall a\, \exists b\, P(a, b)$$

- **Positions of quantifiers can sometimes change**

$$\forall x\, (Q(x) \wedge \exists y\, P(x, y)) \equiv \forall x\, \exists y\, (Q(x) \wedge P(x, y))$$

- **But: order is important...**

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= “x ≥ y”

“There is a number greater than or equal to all numbers.”

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)))$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

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$$\exists x\ \forall y\ \mathrm{GreaterEq}(x, y)))$$

"Every number has a number greater than or equal to it."

$$\forall y\ \exists x\ \mathrm{GreaterEq}(x, y)))$$

x \ y	1	2	3	4
1	T	F	F	F
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3	T	T	T	F
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Predicate Definitions
GreaterEq(x, y) ::= "$x \geq y$"

"There is a number greater than or equal to all numbers."

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)))$$

"Every number has a number greater than or equal to it."

$$\forall y\ \exists x\ \text{GreaterEq}(x, y)))$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Important: both include the case x = y

Different names does not imply different objects!

	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Quantification with Two Variables

expression	when true	when false
$\forall x \forall y\ P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y\ P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y\ P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x\ P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

New Perspective

Rather than comparing **A** and **B** as columns, zooming in on just the rows where **A** is true:

p	*q*	**A**	**B**
T	T	T	
T	F	T	
F	T	F	
F	F	F	

New Perspective

Rather than comparing **A** and **B** as columns, zooming in on just the rows where A is true:

p	q	A	B
T	T	T	T
T	F	T	T
F	T	F	
F	F	F	

Given that A is true, we see that B is also true.

$$A \Rightarrow B$$

New Perspective

Rather than comparing A and B as columns, zooming in on just the rows where A is true:

p	*q*	**A**	**B**
T	T	T	T
T	F	T	T
F	T	F	?
F	F	F	?

When we zoom out, what have we proven?

New Perspective

Rather than comparing **A** and **B** as columns, zooming in on just the rows where B is true:

p	q	A	B	A $\to$ B
T	T	T	T	T
T	F	T	T	T
F	T	F	T	T
F	F	F	F	T

When we zoom out, what have we proven?

$$(A \to B) \equiv \mathbf{T}$$

New Perspective

Equivalences

$A \equiv B$ and $(A \leftrightarrow B) \equiv \mathbf{T}$ are the same

Inference

$A \Rightarrow B$ and $(A \rightarrow B) \equiv \mathbf{T}$ are the same

Can do the inference by zooming in
to the rows where A is true

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with given facts (hypotheses)
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If A and $A \rightarrow B$ are both true, then B must be true
- Write this rule as $$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$$
- Given:
  - If it is Wednesday, then you have a 311 class today.
  - It is Wednesday.
- Therefore, by Modus Ponens:
  - You have a 311 class today.

My First Proof!

Show that r follows from p, p → q, and q → r

1. p Given
2. $p \rightarrow q$ Given
3. $q \rightarrow r$ Given
4.
5.

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

My First Proof!

Show that r follows from p, p → q, and q → r

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

Modus Ponens $\dfrac{A \; ; \; A \rightarrow B}{\therefore \; B}$

Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	Given
2.	$\neg q$	Given
3.	$\neg q \rightarrow \neg p$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

Inference Rules

If A is true and B is true ....

Requirements: $\underline{\quad A \;\; ; \;\; B \quad}$

Conclusions: $\therefore \; C \;\; , \;\; D$

Then, C must be true

Then D must be true

Example (Modus Ponens):

$$\frac{A \;\; ; \;\; A \rightarrow B}{\therefore \qquad B}$$

If I have A and $A \rightarrow B$ both true,
Then B must be true.

Axioms: Special inference rules

If I have nothing...

Requirements:

Conclusions: $\therefore C \;,\; D$

Then, C must be true

Then D must be true

Example (Excluded Middle):

$\therefore A \vee \neg A$

$A \vee \neg A$ must be true.

Simple Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \wedge B}{\therefore A, B}$$

Intro ∧

$$\frac{A \,;\, B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B \,;\, \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Modus Ponens

$$\frac{A \,;\, A \rightarrow B}{\therefore B}$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules