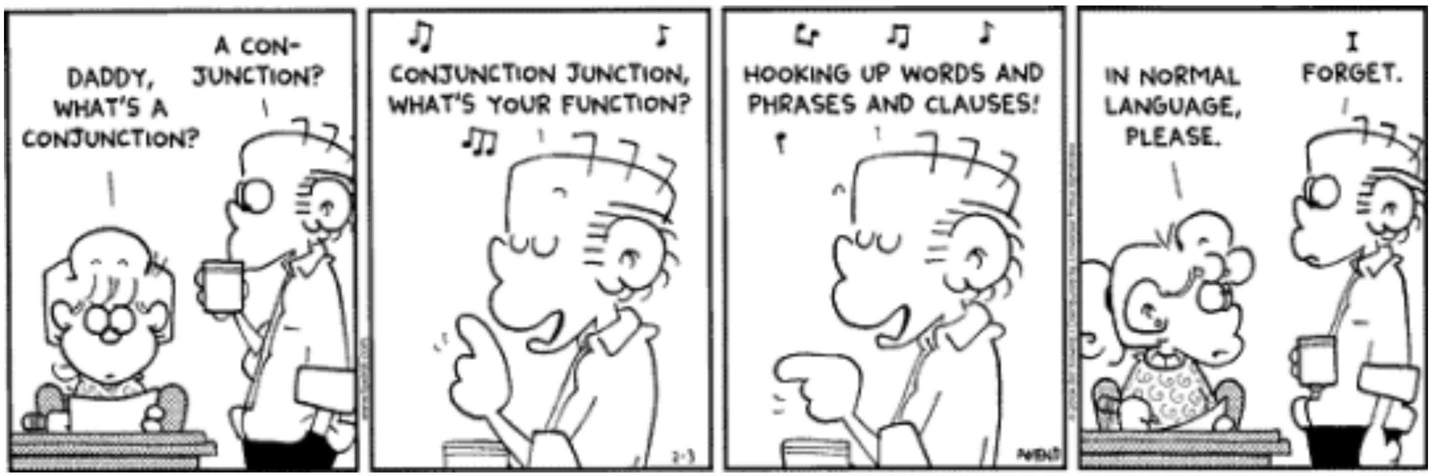
# CSE 311: Foundations of Computing

## Lecture 2: More Logic, Equivalence & Digital Circuits



# Last class: Atomic Propositions

Simplest units (words) in this logical language

Propositional Variables: $p, q, r, s, \ldots$

Truth Values:

- **T** for **true**
- **F** for **false**

# Last class: Some Connectives & Truth Tables

## Negation (not)

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

## Conjunction (and)

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

## Disjunction (or)

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

## Exclusive Or

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

# Last class: Implication

*"If it's raining, then I have my umbrella"*

| $p$ | $r$ | $p \rightarrow r$ |
| --- | --- | --- |
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

In English, we can also write

"I have my umbrella if it's raining."

## $p \rightarrow r$

| $p$ | $r$ | $p \rightarrow r$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Implication:

- *p* implies *r*
- whenever *p* is true *r* must be true
- if *p* then *r*
- *r* if *p*
- *p* only if *r*
- *p* is sufficient for *r*
- *r is necessary for p*

# Biconditional: $p \leftrightarrow r$

- *p* if and only if *r*     (*p* iff *r*)
- *p* implies *r* and *r* implies *p*
- *p* is necessary and sufficient for *r*

| $p$ | $r$ | $p \leftrightarrow r$ |
|---|---|---|
| **T** | **T** | **T** |
| **T** | **F** | **F** |
| **F** | **T** | **F** |
| **F** | **F** | **T** |

# Back to Garfield...

| | |
|---|---|
| $q$ | "Garfield has black stripes" |
| $r$ | "Garfield is an orange cat" |
| $s$ | "Garfield likes lasagna" |

**"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"**

↓

(q if (r and s)) and (r or (not s))

↓

**(q "if"** $(r \wedge s)) \wedge$ **(r** $\vee \neg$**s)**

# Back to Garfield...

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**"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"**

(q if (r and s)) and (r or (not s))

$(\mathbf{q} \text{ “}\mathbf{if}\text{” } (r \land s)) \land (\mathbf{r} \lor \neg\mathbf{s})$

$((r \land s) \longrightarrow \mathbf{q}) \land (\mathbf{r} \lor \neg\mathbf{s})$

# Analyzing the Garfield Sentence with a Truth Table

| $q$ | $r$ | $r$ | $((q \wedge r) \rightarrow p) \wedge (q \vee \neg r)$ |
|---|---|---|---|
| F | F | F | |
| F | F | T | |
| F | T | F | |
| F | T | T | |
| T | F | F | |
| T | F | T | |
| T | T | F | |
| T | T | T | |

# Analyzing the Garfield Sentence with a Truth Table

| $q$ | $r$ | $s$ | $r \vee \neg s$ | $(r \wedge s) \to q$ | $((r \wedge s) \to q) \wedge (r \vee \neg s)$ |
|---|---|---|---|---|---|
| F | F | F | | | |
| F | F | T | | | |
| F | T | F | | | |
| F | T | T | | | |
| T | F | F | | | |
| T | F | T | | | |
| T | T | F | | | |
| T | T | T | | | |

# Analyzing the Garfield Sentence with a Truth Table

| $q$ | $r$ | $s$ | $\neg s$ | $r \vee \neg s$ | $r \wedge s$ | $(r \wedge s) \rightarrow q$ | $((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$ |
|---|---|---|---|---|---|---|---|
| F | F | F | | | | | |
| F | F | T | | | | | |
| F | T | F | | | | | |
| F | T | T | | | | | |
| T | F | F | | | | | |
| T | F | T | | | | | |
| T | T | F | | | | | |
| T | T | T | | | | | |

# Analyzing the Garfield Sentence with a Truth Table

| $q$ | $r$ | $s$ | $\neg s$ | $r \vee \neg s$ | $r \wedge s$ | $(r \wedge s) \rightarrow q$ | $((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$ |
|---|---|---|---|---|---|---|---|
| F | F | F | T | T | F | T | T |
| F | F | T | F | F | F | T | F |
| F | T | F | T | T | F | T | T |
| F | T | T | F | T | T | F | F |
| T | F | F | T | T | F | T | T |
| T | F | T | F | F | F | T | F |
| T | T | F | T | T | F | T | T |
| T | T | T | F | T | T | T | T |

# Converse, Contrapositive

**Implication:**

$$p \rightarrow r$$

**Converse:**

$$r \rightarrow p$$

**Contrapositive:**

$$\neg r \rightarrow \neg p$$

**Inverse:**

$$\neg p \rightarrow \neg r$$

<u>Consider</u>

*p:* *x* is divisible by 2

*r*: *x* is divisible by 4

| | |
|---|---|
| $p \rightarrow r$ | |
| $r \rightarrow p$ | |
| $\neg r \rightarrow \neg p$ | |
| $\neg p \rightarrow \neg r$ | |

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**Consider**

***p:*** ***x*** **is divisible by 2**

***r*****:** ***x*** **is divisible by 4**

| | |
|---|---|
| $p \rightarrow r$ | |
| $r \rightarrow p$ | |
| $\neg r \rightarrow \neg p$ | |
| $\neg p \rightarrow \neg r$ | |

| | Divisible By 2 | Not Divisible By 2 |
|---|---|---|
| Divisible By 4 | | |
| Not Divisible By 4 | | |

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**Consider**

***p:*** ***x*** **is divisible by 2**

***r*****:** ***x*** **is divisible by 4**

| | |
|---|---|
| $p \rightarrow r$ | |
| $r \rightarrow p$ | |
| $\neg r \rightarrow \neg p$ | |
| $\neg p \rightarrow \neg r$ | |

| | Divisible By 2 | Not Divisible By 2 |
|---|---|---|
| Divisible By 4 | 4,8,12,... | Impossible |
| Not Divisible By 4 | 2,6,10,... | 1,3,5,... |

## Converse, Contrapositive

**Implication:**

$p \rightarrow r$

**Converse:**

$r \rightarrow p$

**Contrapositive:**

$\neg r \rightarrow \neg p$

**Inverse:**

$\neg p \rightarrow \neg r$

**Consider**

*p*: *x* is divisible by 2

*r*: *x* is divisible by 4

| | |
|---|---|
| $p \rightarrow r$ | **F** |
| $r \rightarrow p$ | **T** |
| $\neg r \rightarrow \neg p$ | **F** |
| $\neg p \rightarrow \neg r$ | **T** |

| | Divisible By 2 | Not Divisible By 2 |
|---|---|---|
| Divisible By 4 | 4,8,12,... | Impossible |
| Not Divisible By 4 | 2,6,10,... | 1,3,5,... |

# Converse, Contrapositive

**Implication:**

$p \to r$

**Contrapositive:**

$\neg r \to \neg p$

**Converse:**

$r \to p$

**Inverse:**

$\neg p \to \neg r$

**How do these relate to each other?**

| $p$ | $r$ | $p \to r$ | $r \to p$ | $\neg p$ | $\neg r$ | $\neg p \to \neg r$ | $\neg r \to \neg p$ |
|---|---|---|---|---|---|---|---|
| T | T | | | | | | |
| T | F | | | | | | |
| F | T | | | | | | |
| F | F | | | | | | |

# Converse, Contrapositive

**Implication:**

$$p \rightarrow r$$

**Converse:**

$$r \rightarrow p$$

**Contrapositive:**

$$\neg r \rightarrow \neg p$$

**Inverse:**

$$\neg p \rightarrow \neg r$$

An implication and it's contrapositive
have the same truth value!

| $p$ | $r$ | $p \rightarrow r$ | $r \rightarrow p$ | $\neg p$ | $\neg r$ | $\neg p \rightarrow \neg r$ | $\neg r \rightarrow \neg p$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | F | F | T | T |
| T | F | F | T | F | T | T | F |
| F | T | T | F | T | F | F | T |
| F | F | T | T | T | T | T | T |

# Application: Digital Circuits

## Computing With Logic

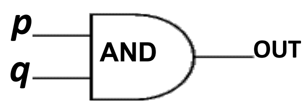
- T corresponds to 1 or "high" voltage
- F corresponds to 0 or "low" voltage

## Gates

- Take inputs and produce outputs (functions)
- Several kinds of gates
- Correspond to propositional connectives (most of them)

# Last class: AND, OR, NOT Gates

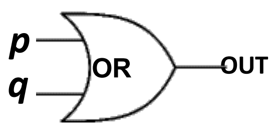
## AND Gate




| $p$ | $q$ | OUT |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

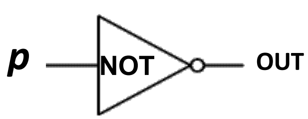
## OR Gate




| $p$ | $q$ | OUT |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

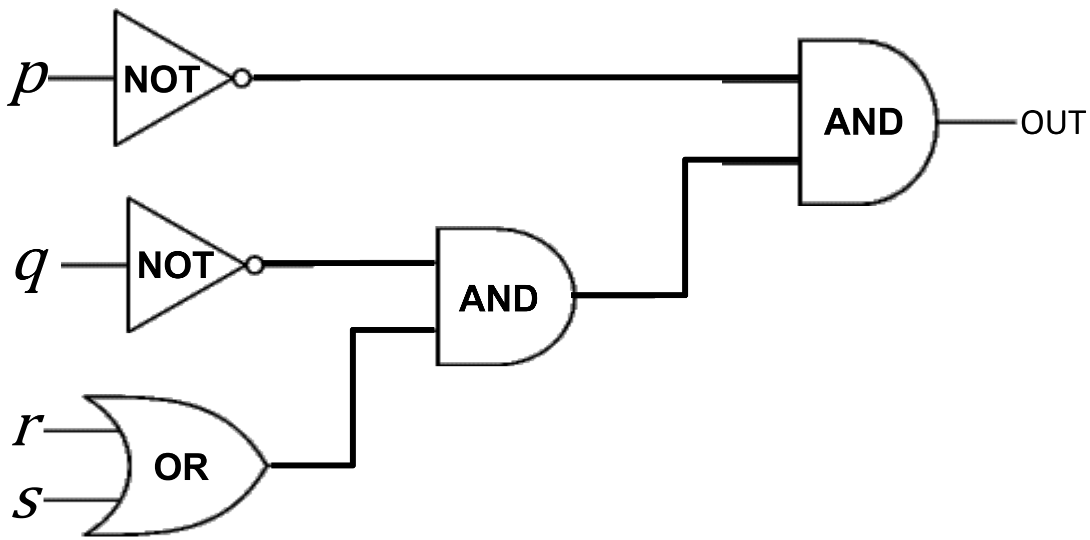
## NOT Gate




| $p$ | OUT |
|---|---|
| 1 | 0 |
| 0 | 1 |

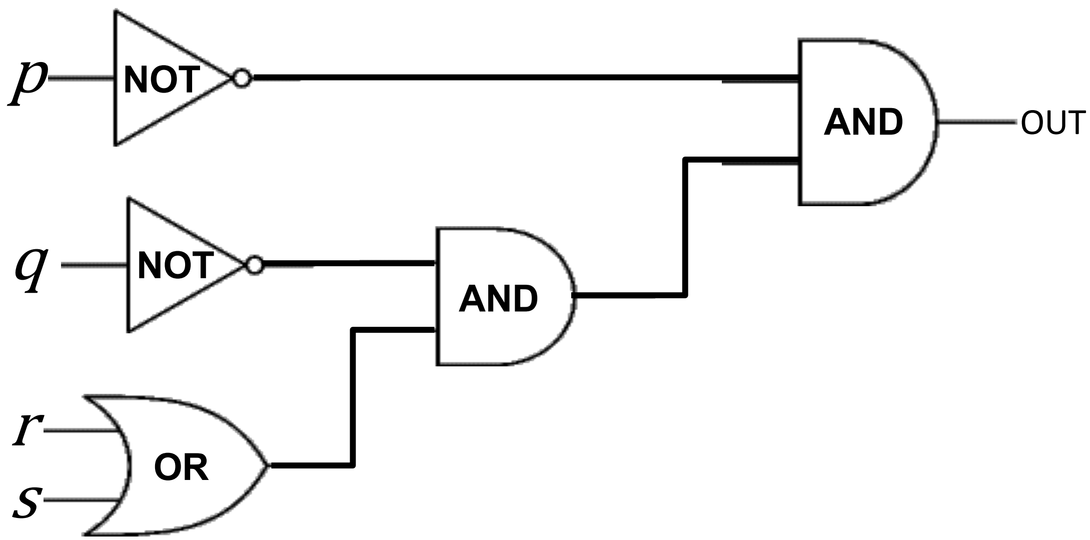
| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

# Combinational Logic Circuits




Values get sent along wires connecting gates

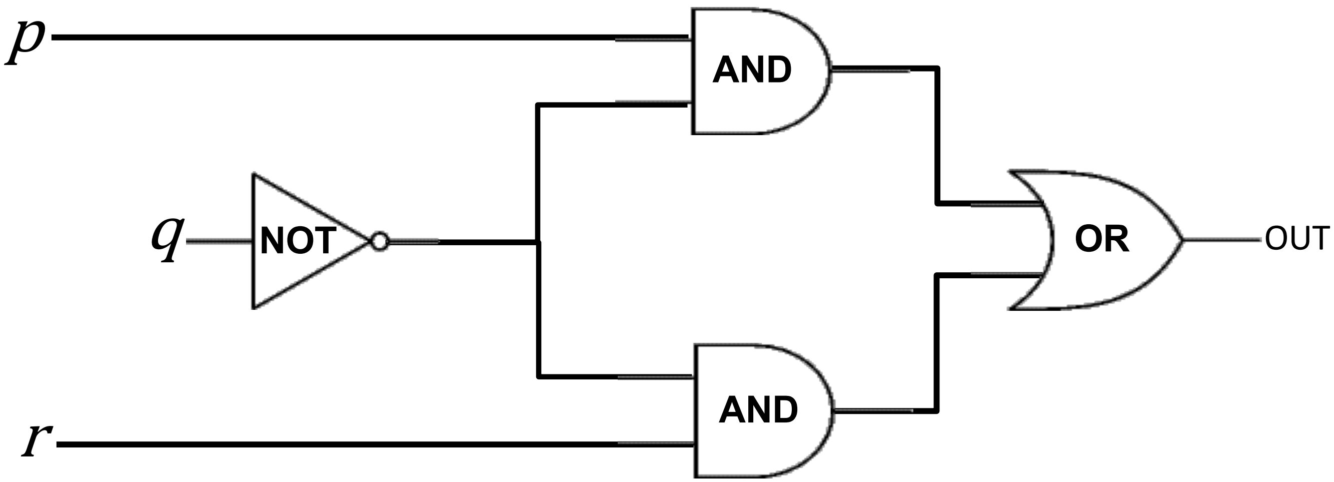
# Combinational Logic Circuits




Values get sent along wires connecting gates

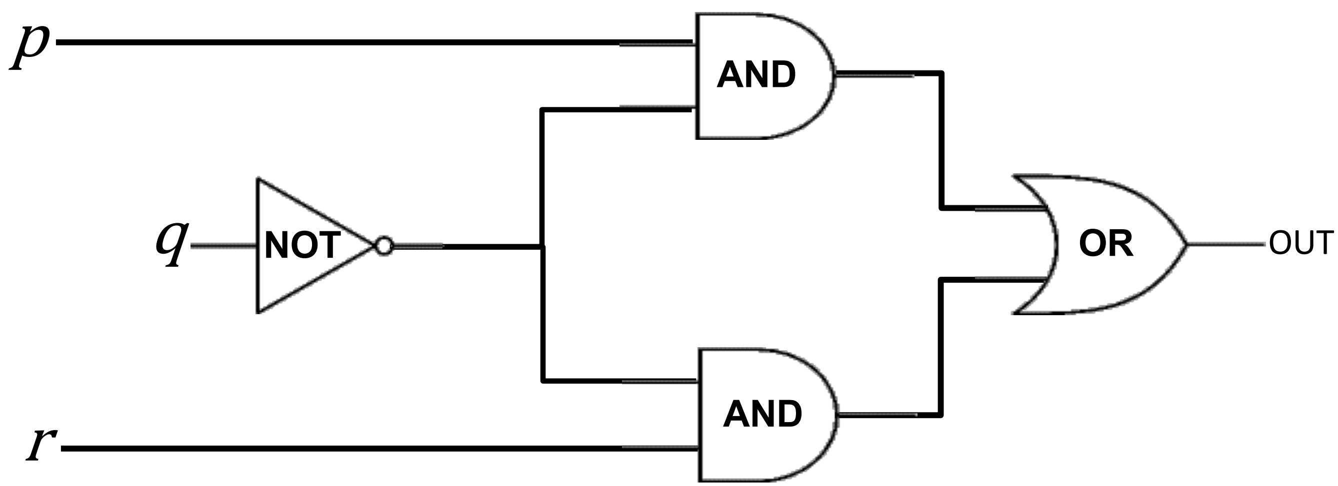
$$\neg p \wedge (\neg q \wedge (r \vee s))$$

# Combinational Logic Circuits




Wires can send one value to multiple gates!

# Combinational Logic Circuits



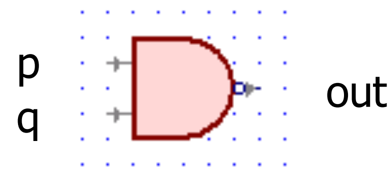

**Wires can send one value to multiple gates!**

$$(p \land \neg q) \lor (\neg q \land r)$$

# Other Useful Gates

**NAND**

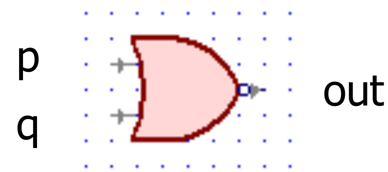
$\neg(p \wedge q)$




| p | q | out |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

**NOR**

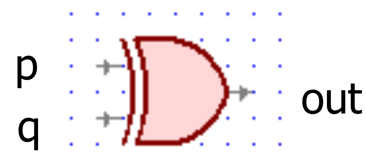
$\neg(p \vee q)$




| p | q | out |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

**XOR**

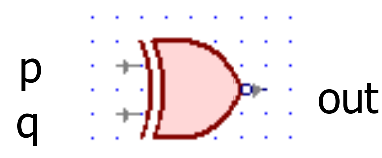
$p \bigoplus q$




| p | q | out |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

**XNOR**

$p \leftrightarrow q$




| p | q | out |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

# Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$

$p \oplus p$

$(p \to r) \wedge p$

# Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$

This is a tautology. It's called the "law of the excluded middle".
If p is true, then $p \vee \neg p$ is true. If p is false, then $p \vee \neg p$ is true.

$p \oplus p$

This is a contradiction. It's always false no matter what truth value p takes on.

$(p \rightarrow r) \wedge p$

This is a contingency. When p=T, r=T, $(T \rightarrow T) \wedge T$ is true.
When p=T, r=F, $(T \rightarrow F) \wedge T$ is false.

# Logical Equivalence

A = B means A and B are identical “strings”:

- $p \wedge r = p \wedge r$

- $p \wedge r \neq r \wedge p$

# Logical Equivalence

**A = B means A and B are identical "strings":**

- $p \wedge r = p \wedge r$

  These are equal, because they are character-for-character identical.

- $p \wedge r \neq r \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

**A $\equiv$ B means A and B have identical truth values:**

- $p \wedge r \equiv p \wedge r$
- $p \wedge r \equiv r \wedge p$
- $p \wedge r \not\equiv r \vee p$

# Logical Equivalence

**A = B means A and B are identical "strings":**

- $p \land r = p \land r$

  These are equal, because they are character-for-character identical.

- $p \land r \neq r \land p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

**A ≡ B means A and B have identical truth values:**

- $p \land r \equiv p \land r$

  Two formulas that are **equal** also are equivalent.

- $p \land r \equiv r \land p$

  These two formulas have the same truth table!

- $p \land r \not\equiv r \lor p$

  When $p$=T and $r$=F, $p \land r$ is false, but $p \lor r$ is true!

## $A \leftrightarrow B$ vs. $A \equiv B$

$A \leftrightarrow B$ is a **proposition** that may be true or false depending on the truth values of $A$ and $B$.

$A \equiv B$ is an **assertion** over all possible truth values that $A$ and $B$ always have the same truth values.

$A \equiv B$ and $(A \leftrightarrow B) \equiv \mathbf{T}$ have the same meaning as does "$A \leftrightarrow B$ is a tautology"

# Logical Equivalence $A \equiv B$

$A \equiv B$ is an assertion that ***two propositions*** A and B always have the same truth values.

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ have the same meaning.

$p \wedge r \equiv r \wedge p$

| $p$ | $r$ | $p \wedge r$ | $r \wedge p$ | $(p \wedge r) \leftrightarrow (r \wedge p)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | F | F | T |
| F | F | F | F | T |