CSE 311: Foundations of Computing I

Homework 5 (due Monday, November 7th at 11:00 PM)

Directions: *Write up carefully argued solutions to the following problems. Each solution should be clear enough that it can explain (to someone who does not already understand the answer) why it works. However, you may use results from lecture, the reference sheets, and previous homeworks without proof.*

1. Fiddler On the Proof (12 points)

Define the following sets.

$$A ::= \{n \in \mathbb{Z} : 6 \mid n\}$$
$$B ::= \{n \in \mathbb{Z} : 2 \mid n\}$$

Now consider the following claim:

$$A \subseteq B$$

(a) [1 Point] What does this claim become if we unroll the definition of "$\subseteq$"?

(b) [8 Points] Write a **formal proof** that the claim from part (a) holds.

(c) [3 Points] Translate the formal proof from (b) into an English proof of the original "$\subseteq$" claim.

2. Feelin' Proovy (8 points)

Let A and B be sets. Consider the following claim:

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

(a) [1 Point] What does this claim become if we unroll the definition of "$=$" sets?

(b) [7 Points] Use the Meta Theorem template to write an **English proof** that the original claim holds.

Note: In your equivalence chain, you can skip steps showing commutativity or associativity, as long as it is easy to follow.

3. Proof, Spoof, or Goof? Set theory edition (12 points)

For each of the claims below, say which of the following categories describes the given proof:

Proof The proof is correct.

Spoof The claim is true but the proof is wrong.

Goof The claim is false.

Then, if it is a spoof, point out the errors in the proof and explain how to correct them, and if it is a goof, point out the *first* error in the proof and then show that the claim is false by giving a counterexample. (If it is a correct proof, then skip this part.)

(a) [4 Points] **Claim**: $(B \cup C) \setminus A \subseteq (B \setminus A) \cap (C \setminus A)$

Proof or Spoof: Let x be arbitrary. Suppose that $x \in (B \cup C) \setminus A$. By the definition of set difference, that means $x \in B \cup C$ and $x \notin A$. By the definition of union, the former means that $x \in B$ and $x \in C$. Putting these facts together again, we have $x \in B \setminus A$ and $x \in C \setminus A$ by the definition of set difference. Since it is in both sets, we have $x \in (B \setminus A) \cap (C \setminus A)$ by the definition of intersection. Since x was arbitrary, we have shown the claimed subset relationship holds.

(b) [4 Points] **Claim**: $A \cap (B \setminus C) \subseteq (A \cup B) \setminus C$

Proof or Spoof: Let x be arbitrary. The definition of intersection tells us that $x \in A \cap (B \setminus C)$ means that $x \in A$ and $x \in B \setminus C$, and the definition of set difference tells us the latter means $x \in B$ and $x \notin C$. Reassembling these facts, $x \in A$ and $x \in B$, together, mean that $x \in A \cup B$, and that, together with the fact that $x \notin C$, means exactly that $x \in (A \cup B) \setminus C$. Hence, x is in $A \cap (B \setminus C)$ iff it is in $(A \cup B) \setminus C$. Since x was arbitrary, we have shown that the two sets are equal.

Finally, the subset relationship is also true for equal sets, and we have shown that the two sets are equal, so the claim holds.

(c) [4 Points] **Claim**: $(\overline{A} \cup \overline{B}) \setminus (A \cap B) = \overline{A \cap B}$

Proof or Spoof: Let x be arbitrary. Then, we can see that the two conditions—being an element of the left set and being an element of the right set—are equivalent as follows:

$$x \in (\overline{A} \cup \overline{B}) \setminus (A \cap B)$$

$\equiv (x \in \overline{A} \cup \overline{B}) \land \neg(x \in A \cap B)$	Def of Difference
$\equiv ((x \in \overline{A}) \lor (x \in \overline{B})) \land \neg(x \in A \cap B)$	Def of Union
$\equiv (\neg(x \in A) \lor \neg(x \in B)) \land \neg(x \in A \cap B)$	Def of Complement
$\equiv \neg((x \in A) \land (x \in B)) \land \neg(x \in A \cap B)$	De Morgan
$\equiv \neg(x \in A \cap B) \land \neg(x \in A \cap B)$	Def of Intersection
$\equiv \neg(x \in A \cap B)$	Idempotence
$\equiv x \in \overline{A \cap B}$	Def of Complement

Since x was arbitrary, we have shown that two sets contain the same elements.

4. Keeping Up With the Cartesians (14 points)

Let A, B, and C be sets. Consider the following claim:

$$A \times B \subseteq B \times C$$

(a) [2 Points] Suppose that $A = \{1, 3\}$, $B = \{1, 3\}$, and $C = \{1, 2, 3\}$.

Calculate the values of the sets $A \times B$ and $B \times C$. Check whether the claim holds for these sets.

(b) [1 Point] What does the claim become if we unroll the definition of "$\subseteq$"?

(c) [8 Points] Write a **formal proof** that the claim from part (b) holds given that $A \subseteq B$ and $B \subseteq C$.

Your proof should use *only* the definitions of $\times$, $\subseteq$, and our rules of inference.

Reminder: $\exists x \in S\,(P(x))$ is shorthand for $\exists x\,((x \in S) \wedge P(x))$. So if you apply Elim $\exists$ with the name a to this statement, the result is $(a \in S) \wedge P(a)$, not just $P(a)$.

(d) [3 Points] Translate the formal proof from (c) into an English proof of the original "$\subseteq$" claim.

5. Our Finest Power (14 points)

Let A and B be sets. Consider the following claim:

$$\mathcal{P}(A) \cap \mathcal{P}(C) \subseteq \mathcal{P}(B \cap C)$$

(a) [2 Points] Suppose that $A = \{1, 2, 3\}$, $B = \{1, 2\}$ and $C = \{2, 3\}$.

Calculate the values of $\mathcal{P}(A) \cap \mathcal{P}(C)$ and $\mathcal{P}(B \cap C)$. Check whether the claim holds for these sets.

(b) [1 Point] What does the claim become if we unroll the definition of "$\subseteq$"? (Be careful!)

(c) [8 Points] Write a **formal proof** that the claim from part (b) holds given that $A \subseteq B$.

Your proof should use *only* the definitions of $\mathcal{P}$, $\cap$, $\subseteq$, and our rules of inference.

Note: Since proving that a set S is an element of $\mathcal{P}(B \cap C)$ requires proving that it is a subset of $B \cap C$, which itself requires proving that *every* element of S is in $B \cap C$ (a $\forall$ claim), your solution to this problem will likely include more nesting levels than your solution to Problem 4(c).

(d) [3 Points] Translate your formal proof from (c) into an English proof of the original "$\subseteq$" claim.

6. Super-Colliding Super Inductor (20 points)

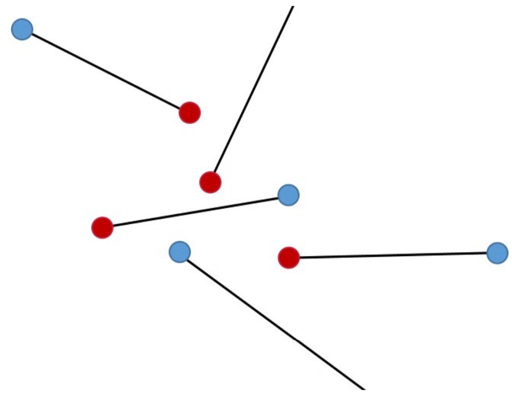
Prove, by induction, that $n^3 + 5n$ is divisible by 3 for any $n \in \mathbb{N}$.

7. Alien Induction (20 points)

Let $x \in \mathbb{R}$ be positive. Prove, by induction, that $(2 + x)^{n+1} > 2^{n+1} + n2^n x$ holds for all $n \in \mathbb{N}$.

8. Extra Credit: Match Me If You Can (0 points)

In this problem, you will show that given n red points and n blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume that there are n red and n blue points fixed in the plane.



A *matching* M is a collection of n line segments connecting distinct red-blue pairs. The *total length* of a matching M is the sum of the lengths of the line segments in M. Say that a matching M is *minimal* if there is no matching with a smaller total length.

Let IsMinimal(M) be the predicate that is true precisely when M is a minimal matching. Let HasCrossing(M) be the predicate that is true precisely when there are two line segments in M that cross each other.

Give an argument in English explaining why there must be at least one matching M so that IsMinimal(M) is true, i.e.

$$\exists M \text{IsMinimal}(M))$$

Give an argument in English explaining why

$$\forall M(\text{HasCrossing}(M) \rightarrow \neg\text{IsMinimal}(M))$$

Then, use the two results above to give a proof of the statement:

$$\exists M \neg\text{HasCrossing}(M).$$