Unalike Nested Quantifiers and New Proof Strategies

First:

- We didn’t quite finish the lecture that was Friday’s. So, please mark on your calendar to:
  - Find the remaining lecture on Canvas under Panopto-> Additional lecture material
  - Take the additional Canvas quiz.

And now:

- A new way of thinking of proofs:

- Here's one way to get an iron-clad guarantee:
- 1. Write down all the facts we know.
- 2. Combine the things we know to derive new facts.
- 3. Continue until what we want to show is a fact.

Drawing Conclusions

- You know "If it is raining, then I have my umbrella"
- And "It is raining"
- You should conclude....

I have my umbrella!

- For whatever you conclude, convert the statement to propositional logic – will your statement hold for any propositions, or is it specific to raining and umbrellas?

I know $(a \to b)$ and a, I can conclude b
Or said another way: $[(a \to b) \land a] \to b$

Modus Ponens

- The inference from the last slide is always valid. I.e.

$$[(a \to b) \land a] \to b \equiv \mathrm{T}$$

Modus Ponens – a formal proof

$[(a \rightarrow b) \wedge a] \rightarrow b$	$\equiv [(\neg a \vee b) \wedge a] \rightarrow b$	Law of Implication
	$\equiv [a \wedge (\neg a \vee b)] \rightarrow b$	Commutativity
	$\equiv [(a \wedge \neg a) \vee (a \wedge b)] \rightarrow b$	Distributivity
	$\equiv [\mathrm{F} \vee (a \wedge b)] \rightarrow b$	Negation
	$\equiv [(a \wedge b) \vee \mathrm{F}] \rightarrow b$	Commutativity
	$\equiv [(a \wedge b)] \rightarrow b$	Identity
	$\equiv [\neg(a \wedge b)] \vee b$	Law of Implication
	$\equiv [\neg a \vee \neg b] \vee b$	DeMorgan's Law
	$\equiv \neg a \vee [\neg b \vee b]$	Associativity
	$\equiv \neg a \vee [b \vee \neg b]$	Commutativity
	$\equiv \neg a \vee \mathrm{T}$	Negation
	$\equiv \mathrm{T}$	Domination

Modus Ponens

- The inference from the last slide is always valid. I.e.

$$[(a \to b) \land a] \to b \equiv \mathrm{T}$$

We use that inference A LOT

So often people gave it a name ("Modus Ponens")

So often…we don't have time to repeat that 12 line proof EVERY TIME.

Let's make this another law we can apply in a single step.

Just like refactoring a method in code.

Notation – Laws of Inference

- We're using the "→ " symbol A LOT.
- Too much

- Some new notation to make our lives easier.

$$\frac{\text{If we know } \textbf{both } A \text{ and } B}{\therefore \quad \text{We can conclude any (or all) of } C, D} \qquad \frac{A, B}{\therefore \quad C, D}$$

"∴" means "therefore" – I knew A, B therefore I can conclude C, D.

$$\frac{a \to b, a}{\therefore \quad b}$$

Modus Ponens, i.e. $[(a \to b) \wedge a] \to b$), in our new notation.

Another Proof

- Let's keep going.
- I know "If it is raining then I have my umbrella" and "I do not have my umbrella"

  It is not raining!
- I can conclude…

- What's the general form? $[(a \rightarrow b) \land \neg b] \rightarrow \neg a$
- How do you think the proof will go?
  - If you had to convince a friend of this claim in English, how would you do it?

A proof!

We know $a \rightarrow b$ and $\neg b$; we want to conclude $\neg a$.
Let's try to prove it. Our goal is to list facts until our goal becomes a fact.
We'll number our facts, and put a justification for each new one.

A proof!

We know $a \to b$ and $\neg b$; we want to conclude $\neg a$.
Let's try to prove it. Our goal is to list facts until our goal becomes a fact.
We'll number our facts, and put a justification for each new one.

1.	$a \to b$	Given
2.	$\neg b$	Given
3.	$\neg b \to \neg a$	Contrapositive of 1.
4.	$\neg a$	Modus Ponens on 3,2.

Try it yourselves

- Suppose you know $a \rightarrow b, \neg s \rightarrow \neg b$,and a. Give an argument to conclude s.

Fill out the poll everywhere for Activity Credit!

Go to pollev.com/cse311 and login with your UW identity
Or text cse311 to 22333

Try it yourselves

- Suppose you know $a \to b, \neg s \to \neg b$,and a. Give an argument to conclude s.

1. $a \to b$ Given
2. $\neg s \to \neg b$ Given
3. a Given
4. b Modus Ponens 1,3
5. $b \to s$ Contrapositive of 2
6. s Modus Ponens 5,4

More Inference Rules

- We need a couple more inference rules.
- These rules set us up to get facts in exactly the right form to apply the really useful rules.
- A lot like commutativity and distributivity in the propositional logic rules.

Eliminate ∧	$A \land B$	I know the fact $A \land B$
	$\therefore \quad A, B$	$\therefore$ I can conclude A is a fact and B is a fact **separately**.

More Inference Rules

- In total, we have two for ∧ and two for ∨, one to create the connector, and one to remove it.

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \wedge B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

- None of these rules are surprising, but they are useful.

The Direct Proof Rule

- **We've been implicitly using another "rule" today, the direct proof rule**

$$\frac{\text{Write a proof "given } A \text{ conclude } B\text{"}}{A \rightarrow B}$$

$$\text{Direct Proof rule} \quad \frac{A \Rightarrow B}{A \rightarrow B}$$

This rule is different from the others – $A \Rightarrow B$ is not a "single fact."
It's an observation that we've done a proof. (i.e. that we showed fact B starting from A.)

We will get a lot of mileage out of this rule…starting next time.

Caution

- Be careful! Logical inference rules can only be applied to **entire** facts. They cannot be applied to portions of a statement (the way our propositional rules could). Why not?
- Suppose we know $a \to b, r$. Can we conclude b?

1.	$a \to b$	Given
2.	r	Given
3.	$(a \vee r) \to b$	Introduce ∨ (1)
4.	$a \vee r$	Introduce ∨ (2)
5.	b	Modus Ponens 3,4.

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

One more Proof

- Show if we know: $a, b, [(a \land b) \to (r \land s)], r \to t$ we can conclude t.

One more Proof

- Show if we know: $a, b, [(a \land b) \to (r \land s)], r \to t$ we can conclude t.

1.	a	Given
2.	b	Given
3.	$[(a \land b) \to (r \land s)]$	Given
4.	$r \to t$	Given
5.	$a \land b$	Intro $\land$ (1,2)
6.	$r \land s$	Modus Ponens (3,5)
7.	r	Eliminate $\land$ (6)
8.	t	Modus Ponens (4,7)

Inference Rules

Eliminate ∧

$$\frac{A \land B}{\therefore \quad A, B}$$

Eliminate ∨

$$\frac{A \lor B, \neg A}{\therefore \quad B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \land B}$$

Intro ∨

$$\frac{A}{\therefore A \lor B, B \lor A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{a \rightarrow b; a}{\therefore \quad b}$$

You can still use all the propositional logic equivalences too!

Unalike Nested Quantifiers

Warm up

Negate the following sentence, and translate both the original and the negation into predicate logic.

Domain of Discourse: Java programs.

If a program throws an exception then it has a bug or received invalid input. (predicates: `ThrowsException, HasBug, BadInput`

Announcements

- Remember to sign up for canvas groups for your lecture breakouts.
- If you don't have a group already, you can join a not-full-one at random.
- We'll try on Friday

- Proof checking tool: https://homes.cs.washington.edu/~kevinz/proof-test/
- Will check your symbolic proofs, so you know if you've applied rules properly. – I do recommend it for rough drafts, I don't recommend for when you're "stuck"

About Grades

- Grades were critical in your lives up until now.
  - If you were in high school, they’re critical for getting into college.
  - If you were at UW applying to CSE, they were key to that application

- Regardless of where you’re going next, what you **learn** in this course matters FAR more than what your grade in this course.
- If you’re planning on industry – interviews matter more than grades.
- If you’re planning on grad school – letters matter most, those are based on doing work outside of class building off what you learned in class.

About Grades

- What that means:
- The TAs and I are going to prioritize your learning over debating whether -2 or -1 is “ more fair”

- If you’re worried about “have I explained enough” – write more!
- It’ll take you longer to write the Ed question than write the extended answer. We don’t take off for too much work.
  - And the extra writing is going to help you learn more anyway.

Regrades

- TAs make mistakes!
- When I was a TA, I made errors on 1 or 2% of my grading that needed to be corrected. If we made a mistake, file a regrade request on gradescope.
- But those are only for mistakes, not for whether "-1 would be more fair"
- If you are confused, please talk to us!
  - My favorite office hours questions are "can we talk about the best way to do something on the homework we just got back?"

  - If **after** you do a regrade request on gradescope, you still think a grading was incorrect, send email to Robbie.
  - Regrade requests will close 2 weeks after homework is returned.

Negation

- Negate these sentences in English and translate the original and negation to predicate logic.
- All cats have nine lives.

$$\forall x(Cat(x) \rightarrow NumLives(x, 9))$$

$\exists x(Cat(x) \land \neg(NumLives(x, 9)))$ "There is a cat without 9 lives."

- All dogs love every person.

$$\forall x \forall y(Dog(x) \land Human(y) \rightarrow Love(x, y))$$

$\exists x \exists y(Dog(x) \land Human(y) \land \neg Love(x, y))$ "There is a dog who does not love someone." "There is a dog and a person such that the dog doesn't love that person."

- There is a cat that loves someone.

$$\exists x \exists y(Cat(x) \land Human(y) \land Love(x, y)$$

$$\forall x \forall y([Cat(x) \land Human(y)] \rightarrow \neg Love(x, y))$$

"For every cat and every human, the cat does not love that human."

"Every cat does not love any human" ("no cat loves any human")

Negation with Domain Restriction

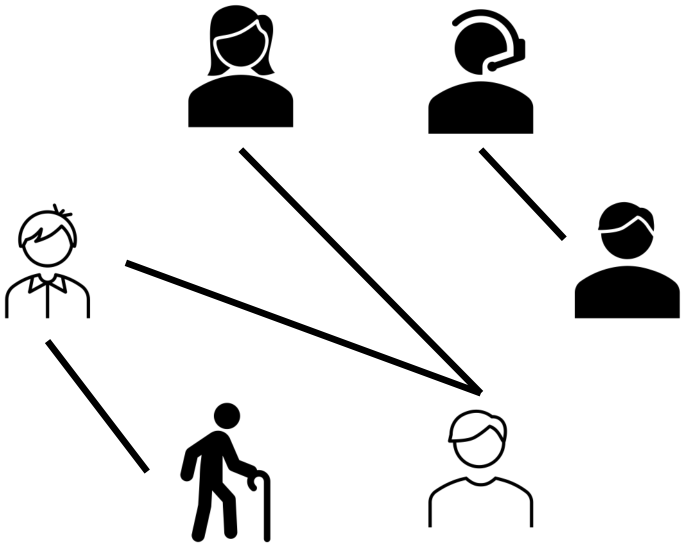
- $\exists x \exists y (Cat(x) \land Human(y) \land Love(x, y)$
- $\forall x \forall y ([Cat(x) \land Human(y)] \rightarrow \neg Love(x, y))$
- There are lots of equivalent expressions to the second. This one is by far the best because it reflects the domain restriction happening. How did we get there?
  - There's a problem in this week's section handout showing similar algebra.

Nested Quantifiers

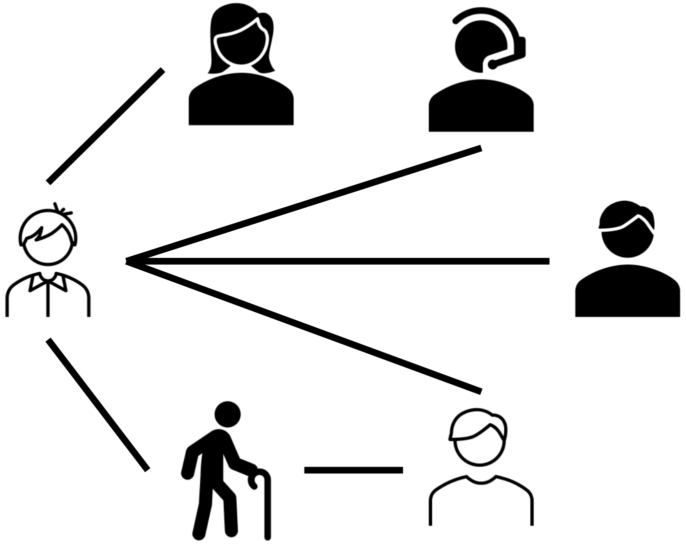
Nested Quantifiers

Translate these sentences using only quantifiers and the predicate `AreFriends`(x, y)

- Everyone is friends with someone.



- Someone is friends with everyone.



Nested Quantifiers

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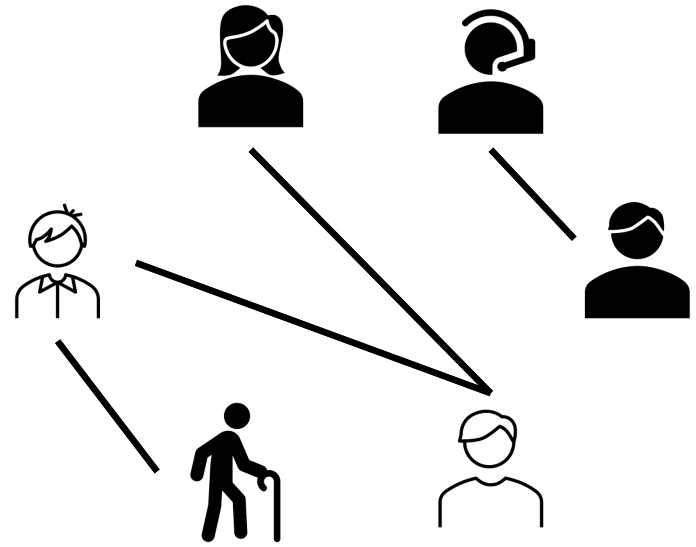

$\forall x(\exists y$ `AreFriends`$(x, y))$

$\forall x \exists y$ `AreFriends`(x, y)

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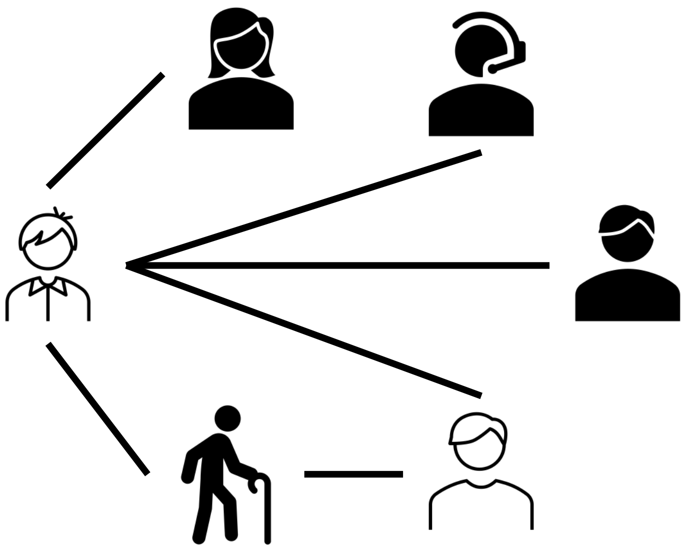


$\exists x(\forall y$ `AreFriends`$(x, y))$

$\exists x \forall y$ `AreFriends`(x, y)

Nested Quantifiers

- $\forall x \exists y\ a(x, y)$
- "For every x there exists a y such that $a(x, y)$ is true."
- y might change depending on the x (people have different friends!).

$\exists x \forall y\ a(x, y)$

"There is an x such that for all $y, a(x, y)$ is true."

There's a special, magical x value so that $a(x, y)$ is true regardless of y.

Nested Quantifiers

- Let our domain of discourse be $\{A, B, C, D, E\}$
- And our proposition $a(x, y)$ be given by the table.
- What should we look for in the table?
- $\exists x \forall y a(x, y)$
- $\forall x \exists y a(x, y)$

y (columns), x (rows)

$a(x, y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Nested Quantifiers

- Let our domain of discourse be $\{A, B, C, D, E\}$
- And our proposition $a(x, y)$ be given by the table.
- What should we look for in the table?
- $\exists x \forall y a(x, y)$
- A row, where every entry is T
- $\forall x \exists y a(x, y)$
- In every row there must be a T

y (columns), x (rows)

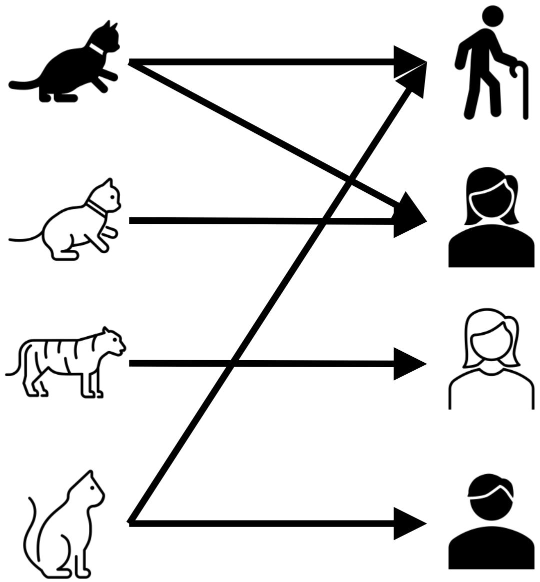
$a(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Keep everything in order

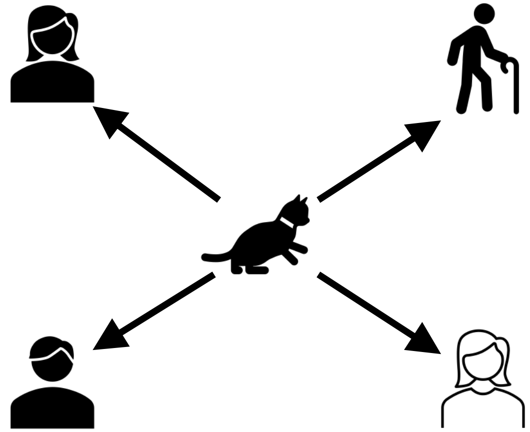
- Keep the quantifiers in the same order in English as they are in the logical notation.
- “There is someone out there for everyone” is a $\forall x \exists y$ statement in “everyday” English.
- It would **never** be phrased that way in “mathematical English” We’ll only every write “for every person, there is someone out there for them.”

Try it yourselves

- Every cat loves some human.



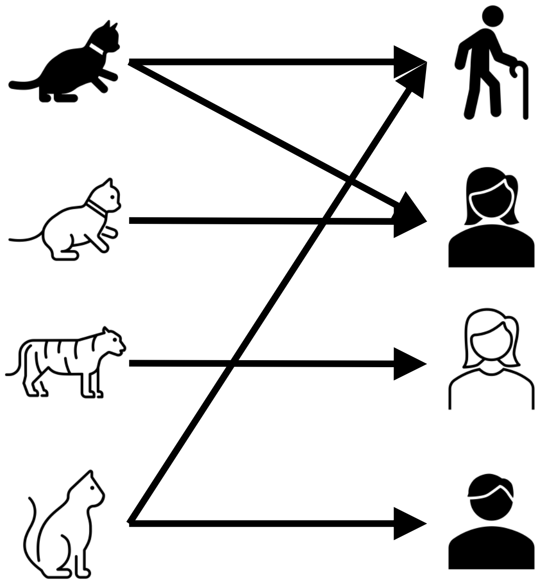
There is a cat that loves every human.



Let your domain of discourse be mammals.
Use the predicates `Cat`(x), `Dog`(x), and `Loves`(x, y) to mean x loves y.

Try it yourselves

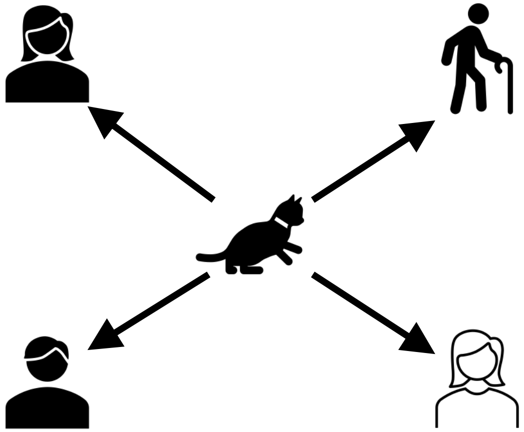
- Every cat loves some human.



$\forall x\ (\mathtt{Cat}(x) \rightarrow \exists y[\mathtt{Human}(y)\ \wedge \mathtt{Loves}(x,y)])$

$\forall x \exists y(\mathtt{Cat}(x) \rightarrow [\mathtt{Human}(y) \wedge \mathtt{Loves}(x,y)])$

There is a cat that loves every human.



$\exists x\ (\mathtt{Cat}(x) \wedge \forall y[\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

$\exists x \forall y(\mathtt{Cat}(x) \wedge [\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

Negation

- How do we negate nested quantifiers?
- The old rule still applies.

To negate an expression with a quantifier
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

$\neg(\forall x \exists y \forall z\ [a(x,y) \wedge b(y,z)])$

$\exists x(\neg(\exists y \forall z\ [a(x,y) \wedge b(y,z)]))$

$\exists x \forall y(\neg(\forall z[a(x,y) \wedge b(y,z)]))$

$\exists x \forall y \exists z(\neg[a(x,y) \wedge b(y,z)])$

$\exists x \forall y \exists z[\neg a(x,y) \vee \neg b(y,z)]$

More Translation

For each of the following, translate it, then say whether the statement is true. Let your domain of discourse be integers.

For every integer, there is a greater integer.

$\forall x \exists y(\texttt{Greater}(y, x))$ (This statement is true: y can be $x + 1$ [y depends on x])

There is an integer x, such that for all integers y, xy is equal to 1.

$\exists x \forall y(\texttt{Equal}(xy, 1))$ (This statement is false: no single value of x can play that role for every y.)

$\forall y \exists x(\texttt{Equal}(x + y, 1))$

For every integer, y, there is an integer x such that $x + y = 1$
(This statement is true, y can depend on x

Inference Proofs and the Direct Proof Rule

Inference Rules

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \wedge B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{a \rightarrow b; a}{\therefore \quad b}$$

You can still use all the propositional logic equivalences too!

How would you argue…

- Let’s say you have a piece of code.
- And you think **if** the code gets null input **then** a nullPointerExecption will be thrown.
- How would you convince your friend?

- You’d probably trace the code, assuming you would get null input.
- The code was your **given**
- **The null input is an assumption**

In general

- How do you convince someone that $a \rightarrow b$ is true given some surrounding context/some surrounding givens?

- You suppose a is true (you assume a)

- And then you'll show b must also be true.
  - Just from a and the Given information.

The Direct Proof Rule

$$\frac{\text{Write a proof "given } A \text{ conclude } B\text{"}}{A \rightarrow B}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

This rule is different from the others – $A \Rightarrow B$ is not a “single fact.”
It’s an observation that we’ve done a proof. (i.e. that we showed fact B starting from A.)

We will get a lot of mileage out of this rule…starting today!

Given: $((a \to b) \land (b \to r))$
Show: $(a \to r)$

- Here's an incorrect proof.

1.	$(a \to b) \land (b \to r)$	Given
2.	$a \to b$	Eliminate $\land$ (1)
3.	$b \to r$	Eliminate $\land$ (1)
4.	a	Given???
5.	b	Modus Ponens 4,2
6.	r	Modus Ponens 5,3
7.	$a \to r$	Direct Proof Rule

Given: $((a \to b) \wedge (b \to r))$
Show: $(a \to r)$

- Here's an incorrect proof.

1. $(a \to b) \wedge (b \to r)$
2. $a \to b$
3. $b \to r$ — Eliminate ∧ (1)
4. a — Given ????
5. b — Modus Ponens 4,2
6. r — Modus Ponens 5,3
7. $a \to r$ — Direct Proof Rule

Proofs are supposed to be lists of facts.
Some of these "facts" aren't really facts…

These facts depend on a.
But a isn't known generally.
It was assumed for the purpose of proving $a \to r$.

Given: $((a \to b) \land (b \to r))$
Show: $(a \to r)$

- Here's an incorrect proof.

1. $(a \to b) \land (b \to r)$
2. $a \to b$
3. $b \to r$ — Eliminate ∧ (1)
4. a — Given ????
5. b — Modus Ponens 4,2
6. r — Modus Ponens 5,3
7. $a \to r$ — Direct Proof Rule

Proofs are supposed to be lists of facts.
Some of these "facts" aren't really facts…

These facts depend on a.
But a isn't known generally.
It was assumed for the purpose of proving $a \to r$.

Given: $(a \rightarrow b) \wedge (b \rightarrow r))$
Show: $(a \rightarrow r)$

- Here's a corrected version of the proof.

1.	$(a \rightarrow b) \wedge (b \rightarrow r)$	Given
2.	$a \rightarrow b$	Eliminate ∧ 1
3.	$b \rightarrow r$	Eliminate ∧ 1
4.1	a	Assumption
4.2	b	Modus Ponens 4.1,2
4.3	r	Modus Ponens 4.2,3
5.	$a \rightarrow r$	Direct Proof Rule

When introducing an assumption to prove an implication: Indent, and change numbering.

When reached your conclusion, use the Direct Proof Rule to observe the implication is a fact.

The conclusion is an unconditional fact (doesn't depend on a) so it goes back up a level

Try it!

- Given: $a \vee b$, $(r \wedge s) \rightarrow \neg b$, Show: $s \rightarrow a$

Eliminate ∧: $\dfrac{A \wedge B}{\therefore\ A, B}$

Eliminate ∨: $\dfrac{A \vee B, \neg A}{\therefore\ B}$

Intro ∧: $\dfrac{A; B}{\therefore\ A \wedge B}$

Intro ∨: $\dfrac{A}{\therefore\ A \vee B, B \vee A}$

Direct Proof rule: $\dfrac{A \Rightarrow B}{A \rightarrow B}$

Modus Ponens: $\dfrac{P \rightarrow Q; P}{\therefore\ Q}$

You can still use all the propositional logic equivalences too!

Try it!

- Given: $a \vee b$, $(r \wedge s) \rightarrow \neg b$, r.
  Show: $s \rightarrow a$

1. $a \vee b$ Given
2. $(r \wedge s) \rightarrow \neg b$ Given
3. r Given
   4.1 s Assumption
   4.2 $r \wedge s$ Intro $\wedge$ (3,4.1)
   4.3 $\neg b$ Modus Ponens (2, 4.2)
   4.4 $b \vee a$ Commutativity (1)
   4.5 a Eliminate $\vee$ (4.4, 4.3)
5. $s \rightarrow a$ Direct Proof Rule