CSE 311 : Practice Midterm

This exam is a (slight) modification of a real midterm given in a prior quarter of CSE311.

The original exam was given in a 50 minute slot; you should expect to spend more time on the Fall 2020 exam (both because you'll be able to write or type more careful solutions than under time pressure and because I'm comfortable giving questions I expect to take slightly longer (since we're not under as much time pressure).

We strongly recommend you take this exam as though it were closed book – even though your exam will be open book.

Instructions

- Students had 50 minutes to complete the exam.
- The exam was closed resource (except for the logical equivalences, boolean algebra, and inference rules reference sheets). You exam will be open resource.
- The problems are of varying difficulty.
- If you get stuck on a problem, move on and come back to it later.

Topic	Max points
1. Translation	20
2. Induction	15
3. Inverses	10
4. Circuits	20
5. [Ir]rational I	5
6. [Ir]rational II	10
7. Squares and mod	20
Total	**100**

1. To Logic...or Not To Logic

1.1. Choose your own predicate adventure [10 points]

(a) Choose a meaning of $P(x, y, z)$ such that $\forall x\ \exists y\ \forall z\ P(x, y, z)$ is false, but $\forall x\ \forall y\ \exists z\ P(x, y, z)$ is true.

(b) In the domain of integers, using any standard mathematical notation (but no new predicates), define $\texttt{Prime}(x)$ to mean "x is prime".

1.2. Games [10 points]

Let the predicates $D(x, y)$ mean "team x defeated team y" and $P(x, y)$ mean "team x has played team y." Give quantified formulas with the following meanings:

(a) Every team has lost at least one game.

(b) There is a team that has beaten every team it has played.

2. Obvious Induction Problem [15 points]

Prove for all $n \in \mathbb{N}$ that the following identity is true:

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

where $x \in \mathbb{R}, x \neq 1$.

3. 311 is Prime! [10 points]

Find all solutions in the range $0 \leq x < 311$ to the modular equation:

$$12x \equiv 5 \pmod{311}$$

4. Even Circuits Are Fun [20 points]

The function `multiple-of-three` takes in two inputs: $(x_1x_0)_2$ and outputs 1 iff $3 \mid (x_1x_0)_2$.

(a) Draw a table of values (e.g. a truth table) for `multiple-of-three`.

(b) Write `multiple-of-three` as a sum-of-products.

(c) Write `multiple-of-three` as a product-of-sums.

(d) Write `multiple-of-three` as a simplified expression (don't bother explaining what rules you're using).

5. Irrationally Rational [5 points]

Recall the definition of irrational is that a number is not rational, and that

$$\texttt{Rational}(x) \equiv \exists p\ \exists q\ x = \frac{p}{q} \land \texttt{Integer}(p) \land \texttt{Integer}(q) \land q \neq 0$$

For this question, you may assume that π is irrational. Disprove that if x and y are irrational, then x + y is irrational.

6. Rationally Irrational [10 points]

Recall the definition of irrational is that a number is not rational, and that

$$\texttt{Rational}(x) \equiv \exists p \, \exists q \; x = \frac{p}{q} \land \texttt{Integer}(p) \land \texttt{Integer}(q) \land q \neq 0$$

Prove that if x and y are rational and $x \neq 7$, then $\dfrac{y^2}{x-7}$ is rational.

7. Gotta $\square$ m $\forall$ [20 points]

We say that k is *a square modulo* m iff there is some integer j such that $k \equiv j^2 \pmod{m}$.
Let $T = \{m \ : \ m = n^2 + 1 \text{ for some integer } n\}$.

(a) Prove that if $m \in T$, then -1 is a square modulo m. (8 points)

(b) Prove that for all integers m and k, if $m \in T$ and k is a square modulo m then $-k$ is also a square modulo m. (12 points)