Notes

- Do not collaborate with anyone besides your prescribed Canvas group.
- The same rules apply as to your homework: you may discuss the problems together, but you must write up your solutions separately, without notes or other artefacts of your discussions.
- You may refer to course materials but no other external materials (no other websites, textbooks, etc). If you unintentionally run into materials relevant to this exam, please cite them.
- The extra credit problem will be worth more substantial points than the usual homework extra credit, worth between \( \frac{1}{2} \) and 1 of the other problems.

1. Induction: a recurrent topic

Consider the following recursively defined function:

\[
T(n) = \begin{cases} 
3 & \text{if } n = 1 \\
6 & \text{if } n = 2 \\
18 & \text{if } n = 3 \\
T(n-3)(n^3 - 3n^2 + 2n) & \text{Otherwise}
\end{cases}
\]

Use induction to prove that that \( T(n) = 3(n!) \) for all \( n \geq 1 \).

2. Putting it all together

Consider the alphabet \( \Sigma = \{0, 1, \diamond\} \).

- Let \( A \) be the set of strings \( x \) from \( \Sigma^* \) such that the following formula holds, where \( \text{CharAt}(x, y, z) \) means that \( x \in \Sigma^* \), \( y \in \mathbb{N} \), \( z \in \Sigma \) and \( z \) is the \( y \)-th character in \( x \).

\[
\forall i \forall j \forall k \forall c \left( \text{CharAt}(x, i, \diamond) \land \text{CharAt}(x, k, c) \land i < k \rightarrow \left(c \neq \diamond \land \left[(i < j \land j < k \land \text{CharAt}(x, j, 1)) \rightarrow c \neq 0 \right]\right) \right)
\]

- Let \( B \) be the set of strings generated by the following CFG.

\[
S \rightarrow A \mid A \diamond M \\
A \rightarrow 0A \mid 1A \mid \varepsilon \\
M \rightarrow 0M \mid M1 \mid MM \mid \varepsilon
\]

- Let \( C \) be the set of strings accepted by the following NFA.

(a) Determine, with proof, whether the sets \( A \) and \( B \) are equal.

(b) Determine, with proof, whether the sets \( A \) and \( C \) are equal.
3. **Structurally Sound**

Prove the regular expression $(3^*1^*3^*)^*$ matches the same language as $(3 \cup 1 \cup 1)^*$, using structural induction on strings over $\Sigma = \{[0 - 9], [A - Z], [a - z]\}$.

4. **Extra credit: Em-POWER-ed Sets**

Let $P(A)$ represent the powerset of set $A$. We can easily argue that $P(\mathbb{N})$ is bijective with $\mathbb{R}$. The proof for this is the standard diagonalization proof for uncountable sets.

Now, suppose I had some set $A$ such that $P(A)$ is bijective with $\mathbb{N}$. Describe this set $A$ or prove that no such set can exist.